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# Trading model with pair pattern strategies

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A simple trading model based on pair pattern strategy space with holding periods is proposed. Power-law behavior is observed for the return variance $\sigma^2$, the price impact $H$ and the predictability $K$ for both models, with linear and square root impact functions. The sum of the traders' wealth displays a positive value for the model with a square root price impact function, and a qualitative explanation is given based on the observation of the conditional excess demand $\langle A|u\rangle$. The cumulative wealth distribution also obeys a power-law behavior with an exponent close to that of real markets. An evolutionary trading model is further proposed. The elimination mechanism effectively changes the behavior of traders, and a power-law behavior is observed in the measure of zero return distribution $P(r = 0)$. The trading model with other types of traders, e.g., traders with the MG's strategies and producers, are also carefully studied.

## 1. Introduction

The standard Minority Game (MG) introduced and studied by Challet and Zhang [1,2] was initially designed as a simplification of Arthur's famous El Farol's Bar problem [3]. It describes a system in which many heterogeneous traders adaptively compete for a scarce resource, and it captures some key features of a generic market mechanism, and the basic interaction between the traders and public information. However, it is a highly simplified model, not suitable for comparing with real financial market trading. To make it more realistic in comparison with the real markets, different variations [4–10] of the standard MG are consequently proposed. For example, the inactive strategy is introduced, which grants traders the possibility of not trading, and thus the number of traders actively trading at each time step varies throughout the game. This type of extension is called the grand canonical MG, and it produces the main characteristics of the stock markets, e.g., the "fat tail" return distribution, and the long-range volatility correlations.

In the standard MG and most of its variations, the strategies give predictions for the next time step, based on the current state of the history, and upon which traders make instantaneous trading actions with a horizon of not more than one time step. For example, the MG-based models with dynamic capitals [5,6], in which the wealth of the traders is updated according to the current trading price, and has no relation with trading they have previously done. The payoff raised from the price change at the next time step is granted to the trader immediately after a single trade. In fact traders have asset holding periods, and make profits from the price difference between two consecutive trading actions of buying and selling in the

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real stock markets. To find a strategy space with holding periods, so that traders can open or close their positions, and holding their positions reasonably is a great challenge for modeling speculation.

Recently, a new model based on a simple pattern-based strategy space with holding periods is proposed by Challet [11]. In this model, the strategy space is composed of a sequence of patterns, i.e., history signals. Traders open or close positions when the current pattern is the pattern listed in the strategy space, and hold positions between patterns. The kind of position he/she might take, i.e., buying or selling is determined by the average price return between two consecutive occurrences of patterns. The explicit trading action of buying or selling is not fixed with the patterns. The simplest case, where the strategy space consists of only one pair of patterns is mainly considered.

Inspired from Challet's work, we introduce a new pattern-based speculation model, in which the patterns strategy space is split into several sub-spaces composed by pairs of patterns. Different from Challet's model, we defined the explicit trading actions of the pair patterns. One strategy consists of a pair of patterns, which denote the history signals for buying and selling. It is reasonable to assume that the traders base their decisions on patterns or history signals, since they may have some experience and know when to buy or sell according to historical signals. The order of the pair patterns does not matter, which means the position can be opened by buying/selling if the historical signal for buying/selling comes first. Therefore, the trader should buy first if he/she wants to sell, or buy only after he/she sells in this model. That is exactly the case in the real stock markets.

The MG is a negative sum game due to the minority essence of its payoffs. Traders compete for limited market resources and only those traders in the minority group are rewarded. Challet's new model keeps the minority-game payoffs impacted by his/her own trade when the trader opens and closes the position. Furthermore, it introduces a new term of majority-game payoffs raised from the contribution of other traders during his/her holding period. The sum of the payoffs therefore depends on the trading frequency and reaction time of each trader, and thus makes the dynamics relatively complicated.

For the real financial market, the sum of the social wealth should be positive due to the general increase of social productivity. Traders are willing to trade in the market which has a positive or at least zero sum of social wealth. The purpose of this paper is to construct a pattern-based model which has a zero or positive wealth sum. We first assume a zero wealth sum in our model by introducing a simple mid-price dynamics with a linear price impact function. Furthermore, a square root impact function, revealed by an empirical study of the real stock market [12–14] is considered, and the model consequently tends to be a positive sum game. The payoff for each trader is naturally determined by the difference between the selling price and the buying price. We name this model as trading model due to the trading essence of the pair patterns.

In Section 2, we first introduce a pattern-based speculation model by Challet, and then introduce our trading model with pair pattern strategies. A comprehensive comparison between the definition of the strategy space and the payoffs of our model and that of Challet's model is presented in detail. Some numerical results of our trading model are subsequently presented compared with Challet's model. In Section 3, a dynamic evolution mechanism is introduced to the trading model, and the process of how traders are washed out, and how their wealth evolves are carefully studied. In Section 4, other types of traders are introduced to the trading model, and Section 5 contains the conclusion.

## 2. Trading model with pattern-based strategy space

### 2.1. A pattern-based speculation model by Challet

The model proposed by Challet [11] consists of N traders, and they base their decisions on patterns. A pattern or history signal records the possible status of the $m$ most recent outcomes of the price change, so there are $P = 2^m$ possible patterns in total. Each trader $i$ is able to recognize $S$ patterns $\mu_{i,1}, \dots, \mu_{i,S}$, drawn randomly and uniformly from all possible $P$ patterns at the beginning of the game and kept fixed throughout the game. Therefore, $P \geq S$. Each trader $i$ keeps track of the cumulative price return between two consecutive occurrences of patterns, denoted by $U_{i,\mu\to\nu}$, where $\mu, \nu \in \mu_{i,1}, \dots, \mu_{i,S}$ and $\mu \neq \nu$. At time $t$, the trader may wish to open their position only when the current history signal $\mu(t)$ is in his/her pattern list, that is, $\mu(t) \in \{\mu_{i,1}, \dots, \mu_{i,S}\}$. The kind of position he/she might open, $a_i(t) = \pm 1$ denoting buying and selling, is determined by average price return between two consecutive occurrences of patterns: if $|U_{i,\mu\to\nu}(t)| > \epsilon t_{\mu\to\nu}$, one buys a share ($U_{i,\mu\to\nu}(t) > 0$) or sells a share ($U_{i,\mu\to\nu}(t) < 0$), where $t_{\mu\to\nu}$ is the total number of time-steps between elapsed patterns $\mu$ and $\nu$ which is a cumulative value over historical records, and $\epsilon > 0$ is a parameter. Then one holds the position until $\mu(t') = \nu$ and closes the position. The excess demand is $A(t) = \sum_{i=1}^{N} a_i(t)$. A linear price impact function is considered, and thus the price return $r(t)$ is simply defined as

$$r(t) = p(t+1) - p(t) = A(t), \tag{1}$$

where $p(t+1)$ is the actual price obtained by the traders who place their orders at time $t$.

Assume that $\mu(t_\mu) = \mu$ and $t_\nu$ is the first subsequent occurrence of pattern $\nu$, the cumulative price return $U_{i,\mu\to\nu}$ between pattern $\mu$ and $\nu$ evolves according to

$$U_{i,\mu\to\nu}(t_\nu + 1) = U_{i,\mu\to\nu}(t_\mu) + p(t_\nu + 1) - p(t_\mu + 1) - (1 - |a_i(t_\mu)|)\zeta_i[A(t_\nu) - A(t_\mu)], \tag{2}$$

where $\zeta_i$ is a naivety factor indicating the reaction time of trader $i$. By adjusting the parameters $\epsilon$ and $\zeta_i$, the waiting time before the traders withdraw from the market is carefully studied.

If trader $i$ decides to open a position at time $t_{i,\mu}$, and close his/her position at time $t_{i,\nu}$, then his/her payoff is

$$a_i[p(t_{i,\nu}+\delta t_{i,\nu})-p(t_{i,\mu}+\delta t_{i,\mu})]=-a_iA(t_{i,\mu},\delta t_{i,\mu})+a_i\sum_{t_{i,\mu}+\delta t_{i,\mu}<t\le t_{i,\nu}}a(t)-(-a_i)A(t_{i,\nu},\delta t_{i,\nu}), \tag{3}$$

$$A(t_{i,\mu},\delta t_{i,\mu})=\sum_{t_{i,\mu}<t\le t_{i,\mu}+\delta t_{i,\mu}}a(t). \tag{4}$$

$\delta t_{i,\mu}$ and $\delta t_{i,\nu}$ are one's reaction time when he/she opens and closes position, which may due to communication delays, and the time needed to make a conscious decision. The first and the last terms are minority-game payoffs, which can be easily recognize by their '$-$' sign: the trader is rewarded if he/she takes an action opposite to the majority of the orders executed during the time delay. The central term which has a '$+$' sign could be regarded as a delayed majority-game payoff: the trader is rewarded if he/she takes an action consequently proved to be consistent with the majority of the orders executed during the holding period. Therefore, the traders' wealth depends on the situation of the market: the relative importance of minority games decreases as the trading frequency decreases, and increases as the reaction time of each trader increases.

## 2.2. *Trading model with pair pattern strategies*

The trading model takes the form of a repeated game with a certain number of traders $N$. Different from challet's model, we split the pattern strategy space into several sub-spaces in units of pair patterns. A strategy consists of a pair of patterns or history signals, with explicit trading actions, labeled as $(\mu,\nu)$, where $\mu$ is for buying and $\nu$ is for selling. A pattern or history signal has the same definition as in Challet's model. Since there is a total of $P=2^m$ possible patterns and the patterns of one strategy should not repeat, there is a total of $2^m(2^m-1)$ probable pairs of patterns.

At the beginning of the game, each trader randomly picks $S'$ strategies from the full strategy space, and keeps them fixed throughout the game, and thus $S'\le 2^m(2^m-1)$. Each trader $i$ keeps track of the cumulative performance of his/her pair pattern strategy $s$, $s=1,\ldots,S'$ by assigning a virtual score $U_{i,s}$ to it. The initial scores of the strategies are set to be zero. At time $t$, each trader $i$ adopts the strategy with the highest score $s_i(t)$, and checks if either of the two patterns of the highest score strategy is consistent with the history at that moment. If the pattern for buying occurs first, the trader opens an position by buying, and holds the position until the pattern for selling occurs. Then the trader closes the position by selling. The trader can also open the position by selling if the pattern for selling occurs first, and then closes the position by buying. Therefore, the model is symmetric. The trader opens and closes the position by taking an action $a_i(t)=\pm 1$, denoting buying and selling. The excess demand is defined as $A(t)=\sum_{i=1}^{N}a_i(t)$.

The price return is defined with a linear price impact function the same as Eq. (1). Assuming that one of the patterns of a strategy occurs at time $t_1$, and $t_2$ is the first subsequent occurrence of the other pattern, the score of the strategy is updated according to the price difference between these two patterns as

$$U_{i,s}(t_2+1)=U_{i,s}(t_1)+p(t_{s_i,\nu}+1)-p(t_{s_i,\mu}+1), \tag{5}$$

where $s_i$ is the strategy of trader $i$. If the pattern for buying $\mu$ occurs first $t_{s_i,\mu}=t_1$ and $t_{s_i,\nu}=t_2$, and if the pattern for selling $\nu$ occurs first $t_{s_i,\nu}=t_1$ and $t_{s_i,\mu}=t_2$. Therefore, the payoff of the strategy is determined by the profit made from the strategy, if it is adopted. In our model, we assume the traders are sophisticated and compute the price return perfectly, and this makes Eq. (5) look like Eq. (2) with $\zeta_i=0$. Wealth $W_i$ is assigned to each trader $i$. If trader $i$ decides to open a position at time $t_1$ and closes it at time $t_2$, their wealth is updated according to the price return between these two trades as

$$W_i(t_2+1)=W_i(t_1)+p(t_{i,\nu}+1)-p(t_{i,\mu}+1). \tag{6}$$

If the trader opens a position by buying first $t_{i,\mu}=t_1$ and $t_{i,\nu}=t_2$, and if the trader opens a position by selling first $t_{i,\nu}=t_1$ and $t_{i,\mu}=t_2$.

However, the model with payoffs defined as above is a negative-sum game, which means that the sum of the wealth of all the traders is negative. Considering the simplest case that the market only has one trader, the payoff of his/her wealth is always $-1$ no matter what kind of position he/she might take first. We assume that the model with linear price impact function has a nature of zero-sum. Inspired by the works in Refs.[8,15], we use a middle price $p(t+\frac{1}{2})=\frac{1}{2}(p(t+1)+p(t))$, where $t+\frac{1}{2}$ is an artificial time, which could be labeled as $t'+1$. Supposing that not all the shares are executed at the price immediately after trading, the traders make trades at the middle price on average. The price return defined with the linear price impact function is

$$r(t')=p(t'+1)-p(t')=\frac{1}{2}(A(t')+A(t'-1)), \tag{7}$$

where $A(t')=A(t)$. The score of the strategy and the wealth of each trader are consequently updated according to the middle price, replacing $t$ by $t'$ in Eqs. (5) and (6).

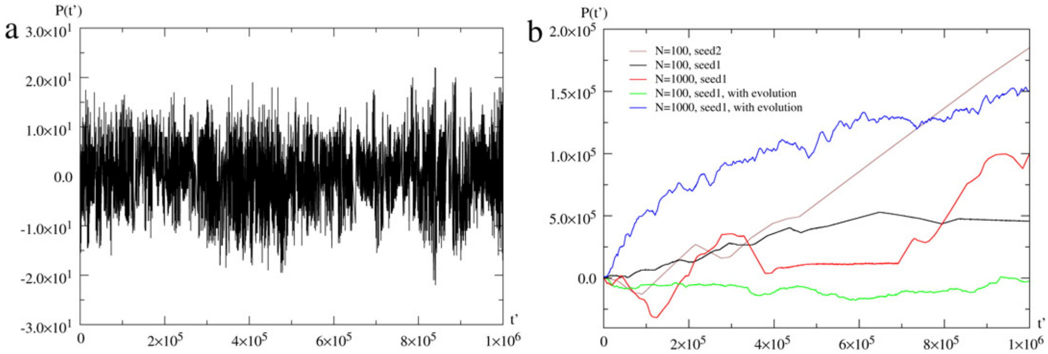


**Fig. 1.** (Color on line) (a) Price evolution of the trading model with linear price impact function at $N = 100$, $S' = 2$ and $m = 3$. (b) Price evolution of the trading model with square root price impact function for different values of the parameter $N$ and different initial seeds at $S' = 2$ and $m = 3$. $10^6$ data are collected after 500 iterations for equilibrium.

Different from Challet's model we simply assume that traders have no reaction time, and thus makes $\delta t = 0$ in Eq. (3). To substitute the price return in Eq. (7) into Eq. (6) and sum over all the traders gives the sum of the traders' wealth at time $t'$

$$\sum_i W_i(t') = \sum_i \sum_{t'_{i,1},t'_{i,2}}^{1\le t'_{i,1},t'_{i,2}\le t'} \left[\frac{1}{2}A(t'_{i,1}) + \frac{1}{2}A(t'_{i,2}) + \sum_{t'_{i,1}<\tau<t'_{i,2}} A(\tau)\right], \tag{8}$$

where $t'_{i,1}$ is the time trader $i$ opens a position and $t'_{i,2}$ is the time trader $i$ closes it. For the model with line price impact function, the sum of the first two terms approximately equals zero, since the number of positions opened by traders equals to that closed by traders. The sum of the traders' wealth consequently depends on the cumulative access demand contributed by the traders during the holding periods.

Recent empirical study shows that the volume imbalance seems to have a square root impact on the price return [12–14]. Therefore, we also introduce a square root impact function to the price return as

$$r(t') = p(t'+1) - p(t') = \frac{1}{2}(\mathrm{sign}(A(t'))\sqrt{|A(t')|} + \mathrm{sign}(A(t'-1))\sqrt{|A(t'-1)|}). \tag{9}$$

The score of the strategy, the wealth of each trader and the sum of the traders' wealth are consequently updated according to this price dynamics.

### *2.3. The results*

In Challet's model, a trader trades between his/her $E$ best pairs of patterns, and therefore $E < \frac{1}{2}S(S-1)$. The simplest case $S = 2$, $E = 1$ is analyzed. We extend the case to $S' = 2$ ($S'$ corresponds to $E$ in Challet's model), so that a trader can trade between two pairs of patterns. We also assume that the trader aims at trading only between his/her most profitable patterns, and therefore $S' < 2^m(2^m - 1)$. Increasing the parameter $m$ will increase the waiting time for the coming patterns, and thus leads to the decreasing number of trading actions. To increase the number of trading actions, we need to increase the parameter $N$. Further increasing the parameter $m$ will not essentially affect the trading model if we increase the parameter $N$ accordingly. Therefore, we fix the parameters $S' = 2$ and $m = 3$, and perform numerical simulations for different values of parameter $N$.

The price evolution of our trading model with linear price impact function for $N = 100$, and square root price impact function for $N = 100$,1000, and different initial seeds are plotted in Fig. 1(a) and (b). For the model with linear price impact function, the curve for $N = 100$ fluctuates symmetrically around zero, and the curve for other value of the parameter $N$ behaves similar to that of $N = 100$ (not shown in figure). For the model with square root price impact function, it seems that the price fluctuates similar to that of the financial markets. However, we observe some attractors for some singular runs, for example the curve for $N = 100$ with *seed*2 changes suddenly close to $t' = 5.0 \times 10^5$ and then the system stuck in a string of periodical history states, which may lead to the abnormal increase of the price.

We first compute the conditional probability $p(u, j)$, which is the conditional probability to have positive, negative and zero price change, denoted by $j = \pm 1, 0$, immediately following a specific history $u$. In Fig. 2(a), (b) and (c), $p(u, j)$ for the trading model with linear price impact function for $N = 50$,100,1000 are plotted. In general, the histograms are not as flat as that of the MG model [16], which means the model has a bias of price change conditional to a specific history. We observe that the histogram for larger $N$ is less flat than the histogram for smaller $N$, which means the price has a relative strong biased tendency at large values of the parameter $N$. In Fig. 3(a), (b) and (c), $p(u, j)$ for the trading model with square root price impact function for $N = 50$,100,1000 are plotted. The curves for the model with square root price impact function are less flat than those of the model with a linear price impact function.

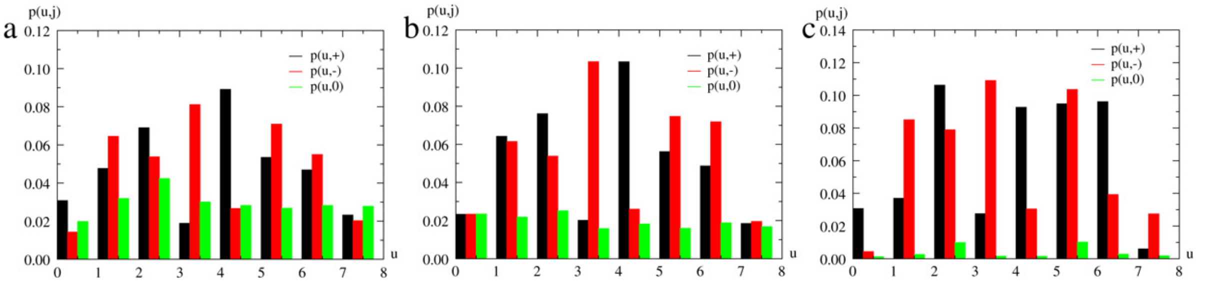


**Fig. 2.** (Color on line) Conditional probability $p(u, j)$ of the model with linear price impact function for: (a) $N = 50$, (b) $N = 100$, (c) $N = 1000$ at $S' = 2$ and $m = 3$.

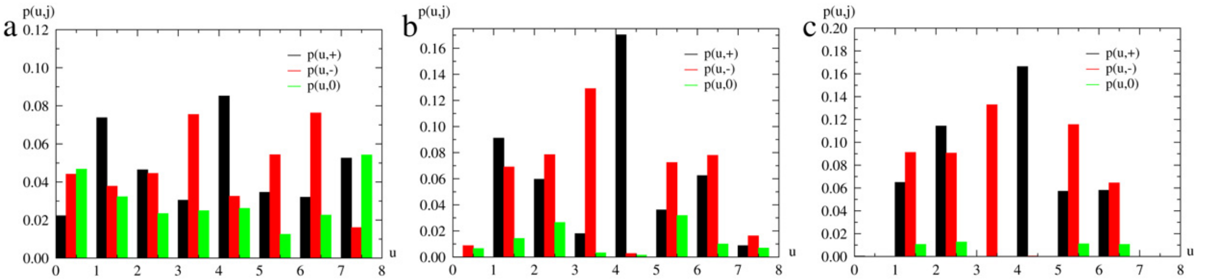


**Fig. 3.** (Color on line) Conditional probability $p(u, j)$ of the model with square root price impact function for: (a) $N = 50$, (b) $N = 100$, (c) $N = 1000$ at $S' = 2$ and $m = 3$.

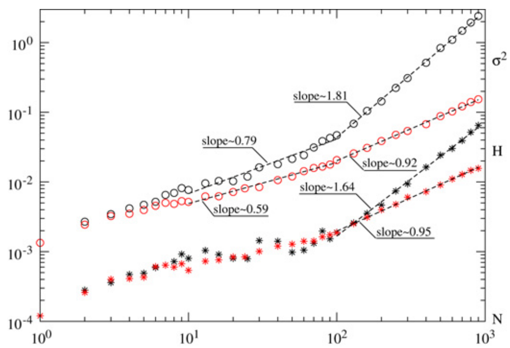


**Fig. 4.** (Color on line) Variance of returns $\sigma^2$ (circles) and price impact $H$ (stars) for the trading models with linear and square root price impact functions at $S' = 2$ and $m = 3$, represented by black and red symbols respectively. The results take average over 100 runs.

The variance of returns is a convenient measure of the market's fluctuation, and it is defined as

$$\sigma^2 = \frac{\langle r^2 \rangle - \langle r \rangle^2}{P}. \tag{10}$$

The smaller $\sigma^2$ is, the less the return fluctuates. In Fig. 4, the variance $\sigma^2$ for the trading models, with linear and square root price impact functions are plotted, denoted by black and red circles respectively. For both models with linear and square root price impact functions, one observes that $\sigma^2$ obeys power-law behavior with a cross-over at approximately $N = 100$, as an order of $2^m(2^m - 1)S'$. $\sigma^2$ for the model with linear price impact function shows a power-law behavior with exponents 0.79 for $N \in [10{,}100)$ and 1.81 for $N \in [100{,}1000)$. $\sigma^2$ for the model with square root price impact function shows a power-law behavior with exponents 0.59 for $N \in [10{,}100)$ and 0.92 for $N \in [100{,}1000)$. Compared with the model with linear price impact function, the model with square root price impact function has a smaller magnitude of price fluctuation, though it has a stronger biased tendency of price change.

To further understand the price return bias conditional to the market states, we compute the average return conditional to a given history defined as

$$H = \frac{\sum_u \langle r|u \rangle^2}{P}. \tag{11}$$

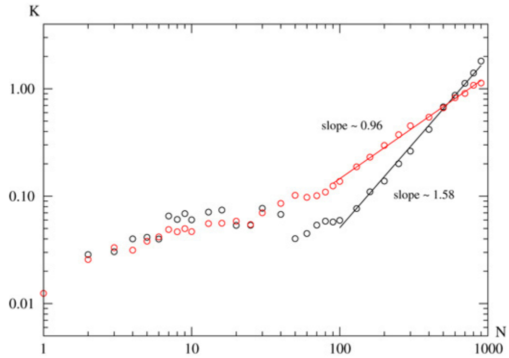


**Fig. 5.** (Color on line) Predictability $K$ for the trading models with linear and square root price impact functions at $S' = 2$ and $m = 3$, represented by black and red circles respectively. The results take average over 100 runs.

In Fig. 4, $H$ for the trading models with linear and square root price impact functions are also plotted, denoted by black and red stars respectively. We observe that $H$ obeys power-law behavior similar to that of $\sigma^2$, with a cross-over at approximately $N = 100$. $H$ for the model with linear price impact function seems constant, and fluctuates slightly for $N \in [10,100)$, then shows a nice power-law behavior with an exponent 1.64 for $N \in [100,1000)$. $H$ for the model with square root price impact function shows a power-law behavior, with an exponent 0.95 for $N \in [100, 1000)$. These two exponents are close to half of the exponents of their price impact functions, which indicates $H$ can be considered as an approximate measure of the average price impact for large values of the parameter $N$.

Another important variable is the predictability that traders hope to exploit from the pair patterns

$$K = \frac{1}{P(P-1)} \sum_{\mu,\nu,\mu\neq\nu} \langle r(t')|\mu \to \nu\rangle^2, \tag{12}$$

where $\langle r(t')|\mu \to \nu\rangle$ stands for the average price return per time step between the occurrence of $\mu$ at time $t'$, and the next occurrence of $\nu$. In Fig. 5, the predictability $K$ for the model with linear and square root price impact functions are plotted, and a cross-over behavior similar to that of $\sigma^2$ and $H$ is observed indicating there maybe exists a phase transition close to the threshold $N = 100$. For the model with a linear price impact function, $K$ increases at the early stage of the parameter $N$, and shows fluctuation for $N \in [10,100)$, then follows a power-law behavior with an exponent 1.58 for $N \in [100,1000)$. For the model with a square root price impact function, $K$ is a monotonously increasing function of the parameter $N$, and shows a power-law behavior with an exponent 0.96 for $N \in [100,1000)$. The model with a square root price impact function is more predictable than the model with a linear price impact function for $N \in [40,400]$, and tends to be less predictable if we further increase the parameter $N$.

In general, cross-over behavior is observed in the measure of $\sigma^2$, $H$ and $K$: for $N \in [10,100)$, $\sigma^2$ increases slowly following a power-law behavior with a relative small exponent, while $H$ and $K$ seems constant and slightly fluctuated; $\sigma^2$, $H$ and $K$ increase rapidly close to $N \sim 2^m(2^m - 1)S' \cong 100$ and tend to follow power-law behavior with relatively big exponents for $N \in [100, 1000)$. For $N > 2^m(2^m - 1)S'$, those traders who have two pairs of pattern strategies completely repeated, make the same trading decisions for buying and selling, and this consequently leads to a rapid increase of $\sigma^2$, $H$ and $K$. A similar cross-over behavior is also observed in Challet's model: different from our trading model, adding more traders first decreases $\sigma^2$, $H$ and $K$. This may because that the model has a fixed number of "producers" who have fixed trading decisions for history $\mu$; then all these quantities reach a minimum at an $N$ of order $P^2$; for $N > \frac{1}{2}P(P-1)$, $\sigma^2$ increases while $H$ and $K$ tend to be constant, and this may because that the traders with same pair of patterns enter and withdraw from the market in a synchronous way. The result of our trading model does not seem to conflict with that of Challet's model.

We then consider the wealth of the traders. The average wealth for each trader $\overline{W} = \frac{\sum_i W_i}{N}$, is calculated. In Fig. 6, the average wealth for each trader for different values of the parameter $N$ at $t' = 10^6$ are plotted. The circles and stars are for the trading models with linear and square root price impact functions, respectively. The average wealth for the model with a linear price impact function is close to zero, independent of the parameter $N$, which indicates that the system is a zero-sum game. Remarkably, the model with a square root price impact displays a positive wealth sum: the curve increases as the increase of the parameter $N$, and can be nicely fitted by a power law with an exponent 0.47 as shown in the insert figure. The exponent is very close to 0.5, which indicates that the average wealth may be equal to the square root of the system size $\overline{W} \sim \sqrt{N}$. This is not difficult to understand, since the price return is supposed to be decided by the square root of the excess demand $A$, which is proportional to the system size $N$.

To understand why the trading model with a linear price impact function displays a zero sum, and the trading model with a square root price impact function displays a positive sum, we further investigate the average excess demand bias $\langle A|u\rangle$ conditional to a specific history $u$. In Fig. 7(a), (b) and (c), $\langle A|u\rangle$ for the trading models with linear and square root price impact functions (represented by black and red circles respectively) for $N = 50, 100, 1000$ are plotted. For the model with

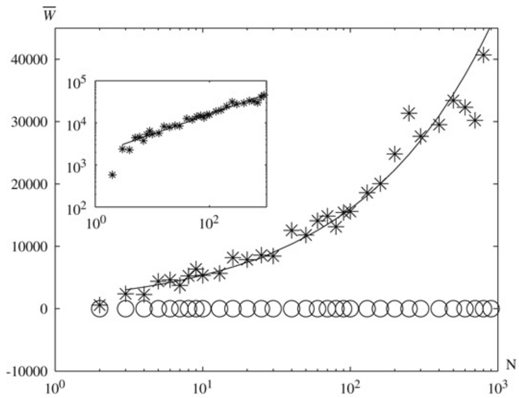


**Fig. 6.** Average wealth $\overline{W}$ for each trader for the trading models with linear and square root price impact functions at $S' = 2$ and $m = 3$, represented by circles and stars respectively. The solid line is the curve fitting $\overline{W} \sim N^{0.47}$. Insert figure is the log–log plot of $\overline{W}$ for the trading model with square root price impact function and the fitting curve.

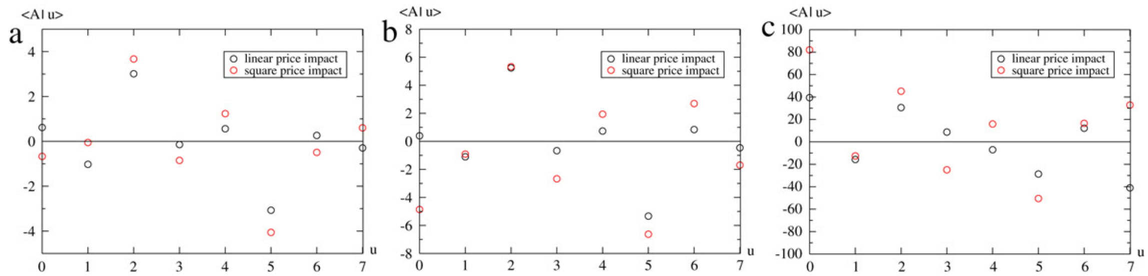


**Fig. 7.** (Color on line) Excess demand bias $\langle A|u\rangle$ for the trading models with linear and square root price impact functions for: (a) $N = 50$, (b) $N = 100$, (c) $N = 1000$ at $S' = 2$ and $m = 3$.

linear price impact function, the sum of the trader's wealth mainly depends on the cumulative access demand contributed by the traders during the holding periods according to Eq. (8). Assuming that the game visits each possible history with a equal probability, so the sum over $\tau$ between two consecutive actions of opening and closing a position could be substituted by the sum over the possible history states between them. $\langle A|u\rangle$ for the trading model with a linear price impact function is symmetrically distributed above and below zero for different states of history $u$. $|\sum_u \langle A|u\rangle|$ displays a value close to zero. Therefore, we observe a zero wealth sum.

The sum of traders' wealth for the model with a square root price impact function mainly depends on the cumulative square root impact of the access demand contributed by the traders during the holding periods, supposing that the sum of the square root impact of the excess demand at which the traders open and close their positions equals zero, i.e., $\sum_i \sum_{t'_{i,1},t'_{i,2}}^{1\le t'_{i,1},t'_{i,2}\le t'} [\frac{1}{2}\mathrm{sign}(t'_{i,1})\sqrt{A(t'_{i,1})} + \frac{1}{2}\mathrm{sign}(A(t'_{i,2}))\sqrt{A(t'_{i,2})}] = 0$, where $t'_{i,1}$ is the time trader $i$ opens a position and $t'_{i,2}$ is the time trader $i$ closes it. $\langle A|u\rangle$ for the model with a square root price impact function is unsymmetrically distributed above and below zero. $|\sum_u \langle A|u\rangle|$ displays a nonzero value obviously larger than that of the model with linear price impact function, e.g., the ratio of $|\sum_u \langle A|u\rangle|$ between two models with square root and linear price impact functions is 7.56, 18.58, 62.66 for $N = 50,100,1000$. The larger the parameter $N$ is, the more unsymmetrical the distribution of $\langle A|u\rangle$ is. Some traders have certain strategies which can help them to effectively exploit the information of the biased excess demand from the patterns, and they make profits from these high-performed strategies. They have wealth greater than zero while other traders have an average wealth close to zero. This may leads to a positive sum for the model with a square root impact function.

In Fig. 8 (a), the wealth distribution of the traders for the model with a square root price impact function for $N = 100$ is plotted according to the rank of their change frequency of the adopted strategies. We observe that the traders who change their strategies more frequently have less wealth. Some traders keep using their high-performed strategies to make more profit, while the others who do not have these strategies always shift among their strategies, and have less wealth. Especially for some singular runs with attractors, we observe that some traders keep using certain high-performed strategies, and the others shift among their strategies at the early stage of the evolution, and eventually withdraw from the market, and thus the system is stuck in a string of period history states. For the model with linear price impact function, the wealth distribution according to the rank of traders' strategy changing frequency shows a similar behavior, but displays a zero sum.

To see the wealth distribution among the traders directly, we study the cumulative probability function of the wealth distribution $P(W) = \int_W^\infty P(w)\mathrm{d}w$, namely the probability to find a person with wealth greater than or equal to $W$. Considerable investigation with real data has revealed that the tail of the income distribution follows a power law

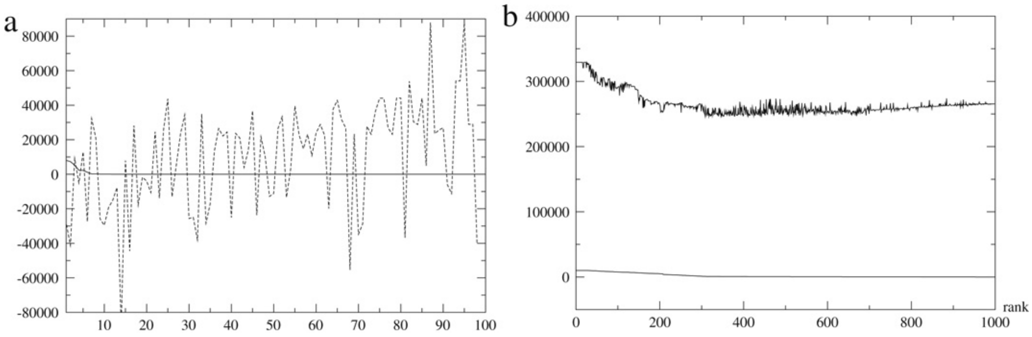


**Fig. 8.** (a) Ranked Wealth distribution of the traders and their corresponding change frequency of the adopted strategies for the model with square root price impact function for $N = 100$ at $S' = 2$ and $m = 3$, represented by dashed and solid lines separately. (b) Ranked wealth distribution of the traders and their corresponding ages for the trading model with evolution for $N = 1000$ at $S' = 2$ and $m = 3$, represented by dashed and solid lines separately.

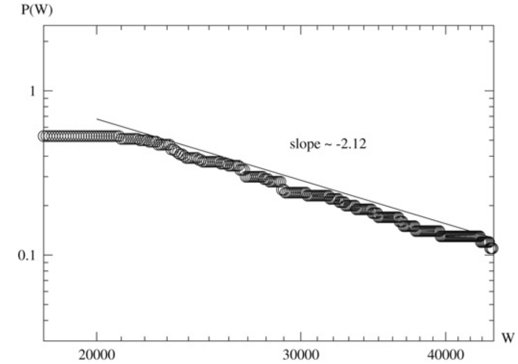


**Fig. 9.** Positive tail of the cumulative wealth distribution $P(W)$ of the trading model with square root price impact function at $S' = 2$, $m = 3$, and $N = 100$. The solid line is the fitting curve $P(W) \sim W^{-2.12}$.

$P(w) \sim w^{-(1+\lambda)}$, where $P(w)$ is the probability density function of the wealth distribution, and $\lambda$ is known as the Pareto index which has a value between 1 and 3 [17–20]. Therefore, the tail of the cumulative wealth distribution follows a power-law behavior $P(W) \sim W^{-\lambda}$. The positive tail of $P(W)$ for the trading model with square root price impact function is shown in Fig. 9. One observes that $P(W)$ obeys a nice power-law behavior in $W \in [2200, 4400]$, and the exponent $\lambda$ is estimated to be 2.12, in agreement with that of the real markets.

### 3. Trading model with evolution

MG-based models with dynamic evolution have been studied in Refs. [6,21,22]. For example, in Refs. [21,22] traders with poor performance can change their strategies. In this trading model, we assume that the worst trader can be driven out of the market following Ref. [6]. Every 100 time steps, the trader who has the lowest wealth is washed out, and a new trader, with new randomly selected strategies, is generated. The wealth of the new trader is set to be the average wealth of the traders at that moment. We use the square root price impact function of the real markets in this evolutionary trading model. With this evolution mechanism, the price evolution becomes more continuous, and seems to be similar to that of the real markets as shown in Fig. 1.

The conditional probability $p(u, j)$ for different values of the parameter $N = 100, 1000$ for this evolutionary trading model are plotted in Fig. 10(a) and (b). It seems that the histograms of the model with evolution are more flat than that of the model without evolution, for the same values of the parameter $N$. The introduction of the elimination mechanism leads to a relatively week biased tendency of the price change. $\sigma^2$, $H$ and $K$ for the evolutionary model are also effectively decreased. In general, the elimination mechanism breaks down the domination of the strategies highly performed in the trading model, and makes the price fluctuations relatively symmetric.

In Fig. 11, the number of the traders still surviving evolved at time $t'$ is plotted. For a large number of traders, e.g., $N = 1000$, those traders who have the strategy (2,5) could survive for quite a long time, and are finally washed out, one after another, in a short time region. There is no high-performed strategy that always keeps winning after we introduce the elimination mechanism. We also observe that the time at which the traders are washed out mainly depends on the parameter $N$. Fig. 12 shows the time $t'$ at which $P_s$ percentage of the traders are washed out as a function of the parameter $N$. $t'$ increases as the increase of the parameter $N$, and obeys a power law, with an exponent close to 1.05; not much different for different values of the parameter $P_s = 25\%, 50\%, 75\%$.

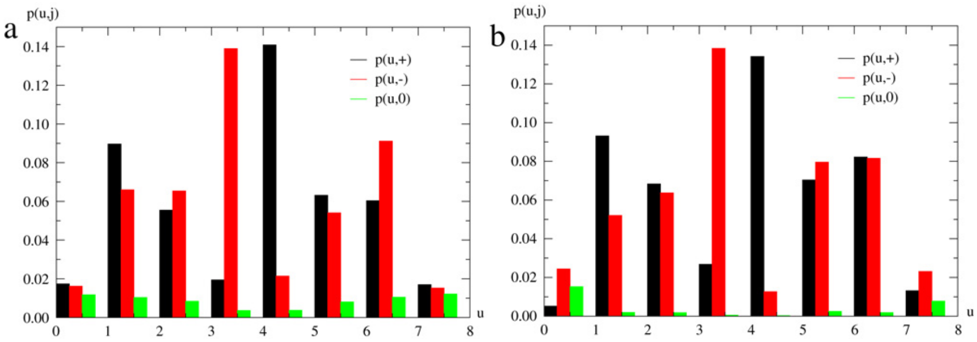


**Fig. 10.** (Color on line) Conditional probability $p(u, j)$ of the trading model with evolution for: (a) $N = 100$, (b)$N = 1000$ at $S' = 2$ and $m = 3$.

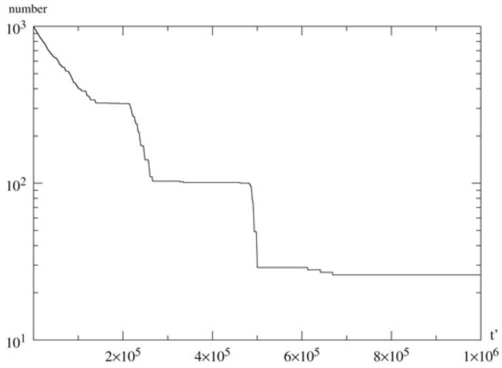


**Fig. 11.** Number of traders still surviving evolves with time $t'$ for the trading model with evolution at $S' = 2$, $m = 3$, and $N = 1000$.

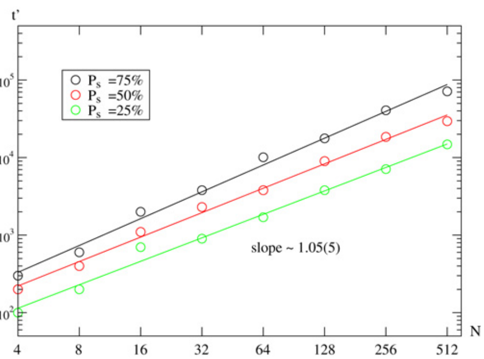


**Fig. 12.** (Color on line) Time $t'$ at which $P_s$ percentage of the traders are washed out for the trading model with evolution at $S' = 2$ and $m = 3$.

Let us take a look at how the traders' wealth evolves before they are washed out. In Fig. 13, the distribution of the relative wealth $(W_i(t') - \overline{W})/\overline{W}$ at different times $t' = 1 \times 10^6, 2 \times 10^6, 2 \times 10^7$ are plotted, where $i$ is the rank of the trader's wealth and $\overline{W}$ is the average wealth of the traders at time $t'$. For $t' = 1 \times 10^6$, the relative wealth is not continuously distributed among the traders. Those traders who have higher wealth are clustered in different groups. For $t' = 2 \times 10^6$, the distribution remains similar, but the relative wealth difference between the rich traders and the poor traders is not so large, and thus the curve becomes more flat. At the time just before all the traders are washed out, e.g., $t' = 2 \times 10^7$, the relative wealth distribution becomes even more flat.

The wealth distribution of the traders ranked by their ages (survival times) for the trading model with evolution, is plotted in Fig. 8 (b). We observe that all the traders have positive wealth due to the evolution mechanism. Those elder traders have relatively more wealth than those younger traders, and those traders newly generated, have an average wealth among them. A power-law-like behavior is also observed in the measure of the cumulative wealth distribution $P(W)$, and it displays a relatively large value, close to the average wealth.

To see whether our trading model can produce the stylized facts of the real financial markets, the return distributions $P(r)$ of the models with evolution for $r = p(t' + \delta t') - P(t')$ with different time windows $\delta t' \geq 1$ are plotted in Fig. 14 (a).

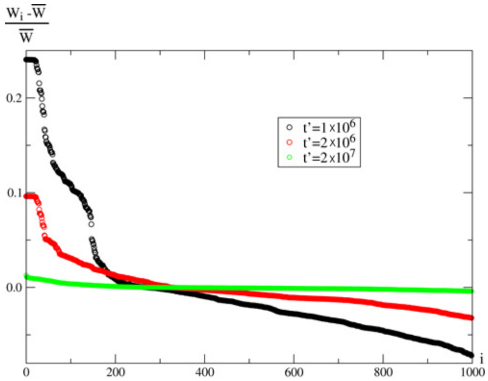


**Fig. 13.** (Color on line) Relative wealth distribution $(W_it' - \overline{W})/\overline{W}$ at different time $t' = 1 \times 10^6, 2 \times 10^6, 2 \times 10^7$ for the trading model with evolution at $S' = 2$, $m = 3$ and $N = 1000$.

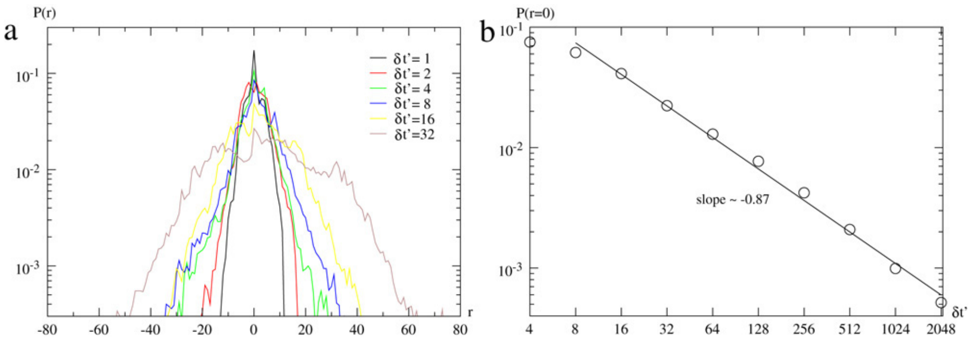


**Fig. 14.** (Color on line) (a) Return distribution $P(r)$ of the trading model with evolution for different time windows $\delta t' \geq 1$ at $S' = 2$, $m = 3$ and $N = 1000$. (b) Zero return distribution $P(r = 0)$ of the trading model with evolution for $S' = 2$, $m = 3$ and $N = 1000$.



Though the return distributions decay exponentially, the curves for bigger $\delta t'$ display tails fatter than those for smaller $\delta t'$. To further understand the behavior of the return distribution, we study the peak values at the center of the return distributions $P(r = 0)$ as the function of different time windows. $P(r = 0)$ is approximated by $\frac{1}{5}\sum_{i=1}^{5}(P(i) + P(-i))/2$, to average over the fluctuation close to zero return. In Fig. 14(b), $P(r = 0)$ of the trading model with evolution is plotted in log–log scale. One observes a power-law behavior with an exponent −0.87, close to −0.71 in real markets [23–26].

## 4. Trading model with other types of traders

We introduce the traders who have the strategies the same as those in the MG model [1,2], which give the predictions for all the probable history status, to the trading model with a square root price impact function, but without evolution. Each of these newly introduced traders has the same number of $S'$ randomly selected strategies, and they trade at each time step, using their best strategies according to the pattern (or history) shared by all the traders. Let $N_t$ and $N_m$ be the number of the traders who have the pair pattern strategies, and who have the strategies the same as those in the MG model. Then the excess demand is defined as $A(t) = \sum_{i=1}^{N_t+N_m} a_i(t)$, and the price dynamics remains the same as Eq. (9). The score of the pair pattern strategy takes the same update form as Eq. (5), and the score of the MG's strategy is updated as $U_{i,s}(t' + 1) = U_{i,s}(t') - a_i(t')(P(t' + 1) - P(t'))$, $i = 1, \ldots, N_m$.

The conditional probability $p(u, j)$, and the wealth distribution of the traders ranked by their change frequency of the adopted strategies for $N_t = 100$ and $N_m = 25$, are plotted in Fig. 15 and Fig. 16(a). The number of the traders effectively trade at each time step $N_{teff} : N_{meff} = 1 : 1$. The histogram of the conditional probability becomes much more flat than that of the model has pure traders who have the pair pattern strategies, and the sum of the wealth of the traders who have the pair pattern strategies tends to be negative. The traders who have the MG's strategies distinctly affect the behavior of the traders who have the pair pattern strategies.

We fix the number of the traders who have the pair pattern strategies $N_t = 100$, and increase the number of the traders who have the MG's strategies, one by one, and see how the traders' wealth behaves. In Fig. 17(a), the average wealth of the traders who have the pair pattern strategies, and the traders who have the MG's strategies for $N_m$ ranging from 1 to 25 at fixed $N_t = 100$ are plotted. For a small $N_m$, the average wealth of the traders who have the MG's strategies is positive, and larger than that of the traders who have pair pattern strategies. A small number of traders dominates the game, and are fed by the majority of traders who have a positive wealth sum. Interestingly, at $N_m \sim 5$ the average wealth of both types of

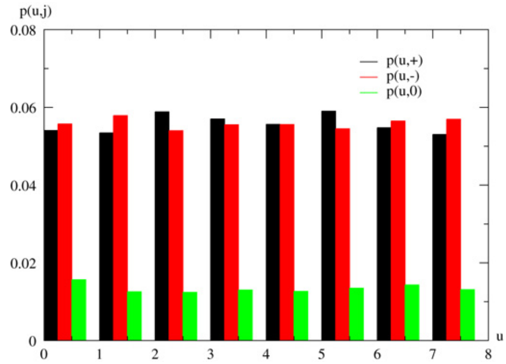


**Fig. 15.** (Color on line) Conditional probability $p(u,j)$ for the trading model with mixed population $N_t = 100$ and $N_m = 25$ at $S' = 2$ and $m = 3$.

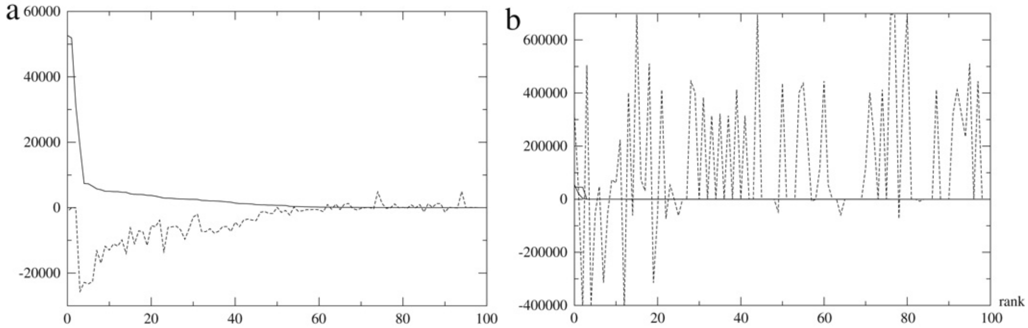


**Fig. 16.** Ranked wealth distribution of the traders who have the pair pattern strategies, and their corresponding change frequency of the adopted strategies for the trading model with mixed population: (a) $N_t = 100$ and $N_m = 25$, (b) $N_t = 100$ and $N_p = 100$ (the number of the traders known as producers) at $S' = 2$ and $m = 3$, represented by dashed and solid lines separately.

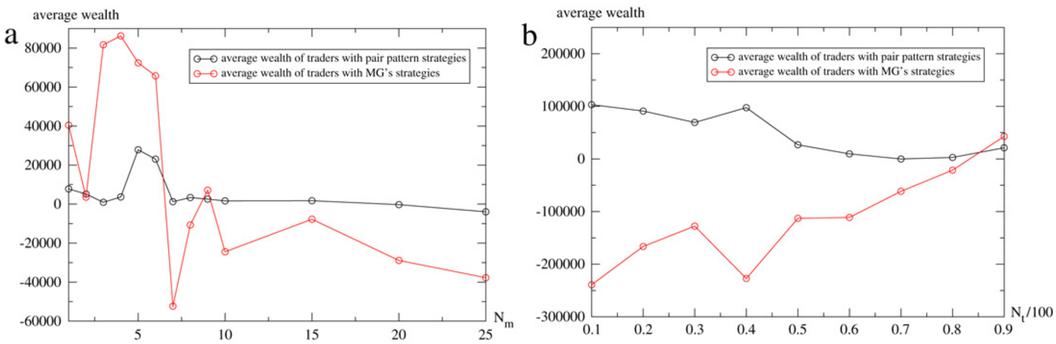


**Fig. 17.** (Color on line) Average wealth of the traders who have the pair pattern strategies, and average wealth of the traders who have the MG's strategies for the trading model with mixed population: (a) $N_m$ ranging from 1 to 25 at fixed $N_t = 100$, (b) different proportion of $N_t/100$ at fixed $N_t + N_m = 100$ for $S' = 2$ and $m = 3$.

trader reaches a maximum. Those two types of trader seem to have a certain state of corporation. As we further increase $N_m$, the average wealth of the traders who have the MG's strategies tends to be negative and smaller than that of the traders who have pair pattern strategies.

Another case, where the total number of the traders is fixed, is also considered, i.e., $N_t + N_m = 100$. We change the proportion of $N_t$ to the total number of the traders. In Fig. 17(b), the average wealth of both types of traders are plotted for fixed $N_t + N_m = 100$. The average wealth of the traders who have the MG's strategies increases with the increase of the proportion of $N_t/100$, while the average wealth of the traders who have the pair pattern strategies decreases with the increase of the proportion of $N_t/100$. For a small $N_t/100$, the average wealth of the traders who have the MG's strategies is negative and smaller than that of the traders who have the pair pattern strategies, and tends to be positive and larger than that of the traders who have the pair pattern strategies for $N_t/100 \sim 0.9$. This is consistent with the result we obtained for fixed $N_t = 100$ shown in Fig. 17(a).

We also introduce the traders known as "producers" in Challet's model to the trading model, with a square root price impact function but without evolution. As it is shown in Fig. 16(b), the introduction of this type of trader can effectively increase the wealth of the traders who have the pair pattern strategies. Most of the traders who have the pair pattern strategies have positive wealth, but the wealth distribution is fluctuates more than the model with the traders who have the MG's strategies shown in Fig. 16(a).

## 5. Conclusion

In summary, a trading model with pair pattern strategies evolved with middle prices is introduced. Both linear and empirical square root price impact functions are considered in the price dynamics, and power-law behaviors are observed for the return variance $\sigma^2$, the price impact $H$, and the predictability $K$ at large values of the parameter $N$. The sum of the traders' wealth displays a positive value for the trading model with a square root price impact function. An unsymmetrical distribution of the conditional excess demand $\langle A|u\rangle$ is observed, and based on this observation, we give a qualitative explanation for the positive wealth sum for the model with a square root price impact function. The cumulative wealth distribution also obeys a power-law behavior, with an exponent 2.12; close to that of real markets. In addition, an evolution mechanism is introduced to the trading model. Newly generated traders with randomly selected strategies break down the domination of the strategies highly performed in the model without evolution, and thus leads to a relatively small value for the biased tendency of the price to change, as well as $\sigma^2$, $H$ and $K$. Power-law behaviors are observed for the time $t'$ at which $P_s$ percentage of the traders are washed out, and the zero return distribution $P(r = 0)$. The traders with the MG's strategies are also introduced to the trading mode. The small fraction of mixed traders are fed by the majority of traders, and thus have more wealth than others. We also introduce the traders known as producers to the trading model, and find that the traders' wealth effectively increases.



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