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Convergence rates of derivatives of a family of barycentric rational interpolants*

Jean-Paul Berrut† Michael S. Floater‡ Georges Klein†

Abstract

In polynomial and spline interpolation the k-th derivative of the interpolant, as a function of the mesh size h, typically converges at the rate of $O(h^{d+1-k})$ as $h \to 0$, where d is the degree of the polynomial or spline. In this paper we establish, in the important cases $k = 1, 2$, the same convergence rate for a recently proposed family of barycentric rational interpolants based on blending polynomial interpolants of degree d.

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1 Introduction

In a recent paper [7], a family of barycentric rational interpolants was investigated. Given a function $f : [a, b] \to \mathbb{R}$ and real values ("nodes") $a = x_0 < x_1 < \cdots < x_n = b$, the d-th interpolant, $d = 0, 1, \ldots, n$, is given by the formula

$$r(x) = \sum_{i=0}^{n-d} \lambda_i(x) p_i(x) \Bigg/ \sum_{i=0}^{n-d} \lambda_i(x), \tag{1}$$

where p_i denotes the polynomial of degree $\le d$ such that $p_i(x_j) = f(x_j)$ for $j = i, i + 1, \ldots, i + d$, and

$$\lambda_i(x) = \frac{(-1)^i}{(x - x_i)(x - x_{i+1}) \cdots (x - x_{i+d})}$$

is a blending function. It was shown in [7] that for $f \in C^{d+2}[a, b]$, the error,

$$e(x) := f(x) - r(x), \tag{2}$$

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†Department of Mathematics, University of Fribourg, Pérolles, 1700 Fribourg, Switzerland, *email:* {*jean-paul.berrut, georges.klein*}*@unifr.ch*

‡Centre of Mathematics for Applications, Department of Informatics, University of Oslo, PO Box 1053 Blindern, 0316 Oslo, Norway, *email: michaelf@ifi.uio.no*

satisfies a bound on the max norm $\|e\| := \max_{a\le x\le b} |e(x)|$ of the form

$$\|e\| \le Ch^{d+1}, \tag{3}$$

where h is the mesh size,

$$h := \max_{0\le i\le n-1} (x_{i+1} - x_i).$$

Here and in all what follows, C is a constant depending only on d, on derivatives of f, and on the interval length $b-a$. Thus, for smooth enough f, the interpolant r converges to f at the rate $O(h^{d+1})$ as $h \to 0$.

A natural, though not definitely settled question is that of a good or even optimal choice of the parameter d for a given function. Through the estimate (3) one might be tempted to think that d should be taken very large; however, the fact that the $p_i(x)$ in (1) are polynomials of degree up to d, interpolating between arbitrary nodes, necessarily restricts d (due to Runge's phenomenon and instability). Nevertheless, numerical experience shows that d may often be chosen quite large. The authors of [10] remark: "This is the only method in this chapter for which we might actually encourage experimentation with high order (say, > 6)". Furthermore, as this family of rational interpolants is linear in the data, the condition of r is determined by its Lebesgue constant Λ_n [4]. Several authors have recently studied Λ_n and shown that it may be bounded for equidistant nodes and $d \ge 2$ as [5]

$$\frac{2^{d-2}}{d+1}\ln(n/d-1) \le \Lambda_n \le 2^{d-1}(2+\ln(n)),$$

thereby confirming that d may be increased to values which yield a very favourable error decay with n (up to about $d = 20$ for large values of n in some cases).

With a view to possible applications such as the numerical solution of differential equations [2], we study here the rate of convergence of derivatives of r to corresponding derivatives of f. Taking into account that r is a blend of polynomial interpolants of degree at most d, it is not unreasonable to expect that

$$\|e^{(k)}\| \le Ch^{d+1-k} \tag{4}$$

for $k = 1, 2, \dots, d$. In this paper we prove that this holds for $k = 1$ and $k = 2$, in some cases under addition of a mesh ratio condition on the nodes x_i; such a condition holds, for example, in the important equally spaced case, $x_i = a + i(b-a)/n$. We conjecture that (4) is valid, at least in the equally spaced case, for all $k = 1, 2, \dots, d$. No fundamental reason hinders the extension of the method to the cases $k \ge 3$; the difficulty is that the formulas for $e^{(k)}$ become very intricate. In Section 2 we will look more closely at the rate of convergence at the nodes, while in Section 3 we extend the theory to intermediate points. The paper ends with numerical examples which confirm the mathematical analysis.

2 Error at the nodes

The Newton error formula

$$f(x) - p_i(x) = (x - x_i)\cdots(x - x_{i+d})[x_i, x_{i+1}, \dots, x_{i+d}, x]f$$

leads to

$$e(x) = \frac{\sum_{i=0}^{n-d} \lambda_i(x)(f(x) - p_i(x))}{\sum_{i=0}^{n-d} \lambda_i(x)} = \frac{A(x)}{B(x)}, \tag{5}$$

where after cancellation

$$A(x) := \sum_{i=0}^{n-d} (-1)^i [x_i, \ldots, x_{i+d}, x] f \tag{6}$$

and

$$B(x) := \sum_{i=0}^{n-d} \lambda_i(x). \tag{7}$$

Consider the first derivative of e at a node x_j, $0 \leq j \leq n$. By the definition of the derivative and the fact that $e(x_j) = 0$, we have

$$e'(x_j) = \lim_{x \to x_j} \frac{e(x)}{x - x_j}.$$

This motivates us to look at the product $(x - x_j)B(x)$. Defining the functions

$$B_j(x) := \sum_{i \in I_j} (-1)^i \prod_{\substack{k=i \\ k \neq j}}^{i+d} \frac{1}{x - x_k} \tag{8}$$

and

$$C_j(x) := \sum_{i \in I \setminus I_j} (-1)^i \prod_{k=i}^{i+d} \frac{1}{x - x_k}, \tag{9}$$

where

$$I = \{0, 1, \ldots, n - d\}$$

and

$$I_j = \{i \in I : j - d \leq i \leq j\},$$

we see that

$$(x - x_j)B(x) = B_j(x) + (x - x_j)C_j(x). \tag{10}$$

Lemma 1

$$e'(x_j) = \frac{A(x_j)}{B_j(x_j)}.$$

Proof. By (10) we have

$$\frac{e(x)}{x - x_j} = \frac{A(x)}{B_j(x) + (x - x_j)C_j(x)}, \tag{11}$$

and taking the limit of both sides as $x \to x_j$ gives the result. □

We can use this formula to obtain an error bound at the nodes that requires f to be in $C^{d+2}[a,b]$, the same as for the bound (3). To increase readability, we introduce the following notation:

$$d_i(x) := |x - x_i|, \quad d_{ik} := |x_i - x_k|,$$

for nodes x_i and x_k and for $x \in [a,b]$, and when it is clear, we also write $d_i = d_i(x)$.

Theorem 1 *If $f \in C^{d+2}[a,b]$, then*

$$|e'(x_j)| \leq Ch^d, \quad 0 \leq j \leq n.$$

Proof. With $x = x_j$ in (8), the products alternate in sign as $(-1)^i$ does, so that all the terms in the sum have the same sign and

$$|B_j(x_j)| = \sum_{i \in I_j} \prod_{\substack{k=i \\ k \neq j}}^{i+d} d_{jk}^{-1}.$$

Therefore, by choosing any $i \in I_j$, we deduce that

$$\frac{1}{|B_j(x_j)|} \leq \prod_{\substack{k=i \\ k \neq j}}^{i+d} d_{jk} \leq Ch^d, \quad \forall\, i \in I_j. \tag{12}$$

On the other hand it has been shown in [7] that

$$|A(x)| \leq C, \quad x \in [a,b], \tag{13}$$

whence the bound follows. □

To deal with higher derivatives, we consider the Taylor expansion

$$e(x) = (x - x_j)e'(x_j) + \frac{1}{2!}(x - x_j)^2 e''(x_j) + \frac{1}{3!}(x - x_j)^3 e'''(x_j) + \cdots \tag{14}$$

and the Taylor expansion of

$$q_j(x) := \frac{e(x)}{x - x_j},$$

namely

$$q_j(x) = q_j(x_j) + (x - x_j)q_j'(x_j) + \frac{1}{2!}(x - x_j)^2 q_j''(x_j) + \cdots .$$

Dividing (14) by $x - x_j$ and comparing terms in the two expansions imply that

$$e^{(k)}(x_j) = k q_j^{(k-1)}(x_j), \tag{15}$$

in particular,

$$e''(x_j) = 2q_j'(x_j). \tag{16}$$

Differentiating (11) and substituting $x = x_j$ give

$$q_j'(x_j) = \frac{A'(x_j)}{B_j(x_j)} - \frac{B_j'(x_j)A(x_j)}{B_j^2(x_j)} - \frac{C_j(x_j)A(x_j)}{B_j^2(x_j)}, \tag{17}$$

which we will use to derive a bound for $e''(x_j)$. We begin with a lemma.

Lemma 2 *If $f \in C^{d+2+k}[a,b]$ for $k \in \mathbb{N}$, then*

$$|A^{(k)}(x)| \le C, \quad x \in [a,b].$$

Proof. The case $k = 0$ has been treated in [7]. For $k \neq 0$ and using the derivative formula for divided differences (see [1] and [8]), we have

$$A^{(k)}(x) = k! \sum_{i=0}^{n-d} (-1)^i [x_i, \ldots, x_{i+d}, (x)^{k+1}]f,$$

where $(x)^k$ stands for a k-fold argument. Then, with a similar approach to [7, p. 322], $A^{(k)}(x)/k!$ equals

$$-\sum_{i=0,\ i \text{ even}}^{n-d-1} (x_{i+d+1} - x_i)[x_i, \ldots, x_{i+d+1}, (x)^{k+1}]f, \quad (n-d) \text{ odd},$$
$$-\sum_{i=0,\ i \text{ even}}^{n-d-2} (x_{i+d+1} - x_i)[x_i, \ldots, x_{i+d+1}, (x)^{k+1}]f + [x_{n-d}, \ldots, x_n, (x)^{k+1}]f, \quad (n-d) \text{ even}.$$

Using the same argument as in [7], we are done. □

Theorem 2 *If $d \ge 1$ and if $f \in C^{d+3}[a,b]$, then*

$$|e''(x_j)| \le Ch^{d-1}.$$

Proof. We write equation (16) with (17) in the form

$$e''(x_j) = 2(L_1 - L_2 - L_3),$$

where

$$L_1 := \frac{A'(x_j)}{B_j(x_j)}, \quad L_2 := \frac{B_j'(x_j)A(x_j)}{B_j^2(x_j)}, \quad L_3 := \frac{C_j(x_j)A(x_j)}{B_j^2(x_j)},$$

and we show that

$$|L_1| \le Ch^d, \tag{18}$$

and

$$|L_2|,\ |L_3| \le Ch^{d-1}. \tag{19}$$

Equation (18) immediately follows from equation (12) and Lemma 2. To deal with L_2, we notice that

$$B_j'(x) = \sum_{i\in I_j}(-1)^{i+1}\sum_{\substack{m=i\\ m\neq j}}^{i+d}\frac{1}{x-x_m}\prod_{\substack{k=i\\ k\neq j}}^{i+d}\frac{1}{x-x_k},$$

so that its absolute value in $x = x_j$ is bounded by

$$|B_j'(x_j)| \leq \sum_{i\in I_j}\sum_{\substack{m=i\\ m\neq j}}^{i+d} d_{jm}^{-1}\prod_{\substack{k=i\\ k\neq j}}^{i+d} d_{jk}^{-1}. \tag{20}$$

To derive a bound of the quotient of (20) with $|B_j(x_j)|^2$, we use (12) with $i \in I_j$ equal to the index in the outer sum in (20) :

$$\frac{|B_j'(x_j)|}{|B_j(x_j)|^2} \leq \sum_{i\in I_j}\sum_{\substack{m=i\\ m\neq j}}^{i+d} d_{jm}^{-1}\prod_{\substack{k=i\\ k\neq j}}^{i+d} d_{jk}^{2}\prod_{\substack{k=i\\ k\neq j}}^{i+d} d_{jk}^{-1} \leq \sum_{i\in I_j}\sum_{\substack{m=i\\ m\neq j}}^{i+d}\prod_{\substack{k=i\\ k\neq j\\ k\neq m}}^{i+d} d_{jk} \leq Ch^{d-1}$$

and this, together with (13), gives the bound on L_2 in (19).

Finally, we treat L_3. We split $C_j(x_j)$ into two parts,

$$C_j(x_j) = \sum_{i=0}^{j-d-1}(-1)^i\prod_{k=i}^{i+d}\frac{1}{x_j-x_k} + \sum_{i=j+1}^{n-d}(-1)^i\prod_{k=i}^{i+d}\frac{1}{x_j-x_k},$$

where empty sums are meant to equal 0. The terms in both sums are alternating in sign and increasing, respectively decreasing, in absolute value, so that

$$|C_j(x_j)| \leq \prod_{k=j-d-1}^{j-1} d_{jk}^{-1} + \prod_{k=j+1}^{j+1+d} d_{jk}^{-1}. \tag{21}$$

We now divide every term in equation (21) by $|B_j(x_j)|^2$. Using (12) with $i = j-d$ for the first term and $i = j$ for the second, we obtain

$$\frac{|C_j(x_j)|}{|B_j(x_j)|^2} \leq \frac{\prod_{k=j-d}^{j-1} d_{jk}^2}{\prod_{k=j-d-1}^{j-1} d_{jk}} + \frac{\prod_{k=j+1}^{j+d} d_{jk}^2}{\prod_{k=j+1}^{j+d+1} d_{jk}} = \frac{\prod_{k=j-d}^{j-1} d_{jk}}{d_{j,j-d-1}} + \frac{\prod_{k=j+1}^{j+d} d_{jk}}{d_{j,j+d+1}}.$$

Since

$$\frac{d_{j,j-d}}{d_{j,j-d-1}} \leq 1, \quad \frac{d_{j,j+d}}{d_{j,j+d+1}} \leq 1,$$

it follows that

$$\frac{|C_j(x_j)|}{|B_j(x_j)|^2} \leq \prod_{k=j-d+1}^{j-1} d_{jk} + \prod_{k=j+1}^{j+d-1} d_{jk} \leq Ch^{d-1},$$

which, together with (13), gives the bound on L_3 in (19). □

3 Error at intermediate points

We now consider the rate of convergence of the derivative of the error at any point x in $[a, b]$. For the first derivative we obtain the same rate of convergence as at the nodes, namely $O(h^d)$, but only under the stricter condition that $f \in C^{d+3}[a, b]$.

Theorem 3 *If $d \geq 2$ and if $f \in C^{d+3}[a, b]$, then*

$$\|e'\| \leq Ch^d.$$

Proof. Due to the continuity of e', it is sufficient to let $x \in (x_j, x_{j+1})$ and to show that

$$|e'(x)| \leq Ch^d, \tag{22}$$

independently of j. To establish (22), we differentiate (5), to obtain

$$e'(x) = \frac{A'(x)}{B(x)} - A(x)\frac{B'(x)}{B^2(x)}. \tag{23}$$

In the proof of Theorem 2 of [7] it was shown that

$$|B(x)| \geq \frac{1}{Ch^{d+1}}, \quad \forall x \in [a, b],$$

and so, from Lemma 2, it follows that

$$\frac{|A'(x)|}{|B(x)|} \leq Ch^{d+1}.$$

Since $|A(x)| \leq C$, it remains to show that

$$\frac{|B'(x)|}{|B^2(x)|} \leq Ch^d. \tag{24}$$

We use the following index sets introduced in [7] and which subdivide I:

$$J_1 = \{i \in I : i \leq j - d\}, \quad J_2 = \{i \in I : j - d + 1 \leq i \leq j\}, \quad J_3 = \{i \in I : j + 1 \leq n - d\}.$$

Now

$$|B'(x)| = \left| \sum_{i \in I} \sum_{m=0}^{d} \frac{(-1)^{i+1}}{(x - x_i) \cdots (x - x_{i+d})(x - x_{i+m})} \right| \leq \sum_{m=0}^{d} (M_{m,1} + M_{m,2} + M_{m,3}), \tag{25}$$

where we have interchanged the summation order and set

$$M_{m,p} := \left| \sum_{i \in J_p} \frac{(-1)^{i+1}}{(x - x_i) \cdots (x - x_{i+d})(x - x_{i+m})} \right|, \quad p = 1, 2, 3.$$

If $J_p = \varnothing$, then $M_{m,p} = 0$. J_2 is not empty since $d \geq 2$.
For every fixed m, the terms in the sums in $M_{m,1}$ and $M_{m,3}$ are alternating in sign and increasing, respectively decreasing, in absolute value and so

$$M_{m,1} \leq \frac{1}{d_{j-d} \cdots d_j d_{j-d+m}} \quad \text{and} \quad M_{m,3} \leq \frac{1}{d_{j+1} \cdots d_{j+1+d} d_{j+1+m}}.$$

In the same proof in [7, p. 322], it has been shown that

$$|B(x)| \geq |\lambda_i(x)|, \quad \forall\, i \in J_2. \tag{26}$$

Next, we divide $M_{m,1}$ by $|B(x)|^2$ and use (26) with $i = j - d + 1$, so that

$$\frac{M_{m,1}}{|B(x)|^2} \leq \frac{d_{j-d+1}^2 \cdots d_{j+1}^2}{d_{j-d} \cdots d_j d_{j-d+m}} = \frac{d_{j-d+1} \cdots d_j d_{j+1}^2}{d_{j-d} d_{j-d+m}},$$

and since $d_j / d_{j-d+m} \leq 1$ for $m = 0, \ldots, d$ and $d_{j-d+1}/d_{j-d} \leq 1$:

$$\frac{M_{m,1}}{|B(x)|^2} \leq d_{j-d+2} \cdots d_{j-1} d_{j+1}^2 \leq Ch^d.$$

Similarly

$$\frac{M_{m,3}}{|B(x)|^2} \leq Ch^d.$$

Finally we bound $M_{m,2}/|B(x)|^2$. Choosing the same $i \in J_2$ in (26) as in each term of the sum in $M_{m,2}$, it follows

$$\frac{M_{m,2}}{|B(x)|^2} \leq \sum_{i \in I_2} \frac{d_i^2 \cdots d_{i+d}^2}{d_i \cdots d_{i+d} d_{i+m}} = \sum_{i \in I_2} d_i \cdots d_{i+m-1} d_{i+m+1} \cdots d_{i+d} \leq Ch^d.$$

Thus (24) follows from (25). $\square$

In the case $d = 1$, we obtain the same rate of convergence, $O(h)$, as for the larger d in Theorem 3 but only under a bounded mesh ratio.

Theorem 4 *If $d = 1$ and if $f \in C^4[a, b]$, then*

$$\|e'\| \leq C(2\beta + 1)h,$$

where

$$\beta := \max\left\{ \max_{1 \leq i \leq n-1} \frac{d_{i,i+1}}{d_{i,i-1}}, \max_{0 \leq i \leq n-2} \frac{d_{i+1,i}}{d_{i+1,i+2}} \right\}.$$

Proof. Again we determine the open subinterval (x_j, x_{j+1}) containing x and consider (23). Since the bounds for $|A(x)|$, $|A'(x)|$ and $|B(x)|$ from the previous theorem also hold for $d = 1$, we bound $|B'(x)|/|B(x)|^2$ for $d = 1$ and $J_2 = \{j\}$. Using similar arguments as in that theorem, we obtain

$$\begin{aligned}\frac{|B'(x)|}{|B(x)|^2} &\leq \sum_{m=0}^{1}\left(\frac{1}{|B(x)|^2}\left|\sum_{i\in I}\frac{(-1)^{i+1}}{(x-x_i)(x-x_{i+1})(x-x_{i+m})}\right|\right)\\ &\leq \sum_{m=0}^{1}\left(\frac{d_j^2 d_{j+1}^2}{d_{j-1}d_j d_{j-1+m}} + \frac{d_j^2 d_{j+1}^2}{d_j d_{j+1} d_{j+m}} + \frac{d_j^2 d_{j+1}^2}{d_{j+1}d_{j+2}d_{j+1+m}}\right)\\ &\leq 2\frac{d_{j+1}^2}{d_{j-1}} + d_{j+1} + d_j + 2\frac{d_j^2}{d_{j+2}}\\ &\leq 2(2\beta+1)h.\end{aligned}$$

□

For the second derivative the mesh ratio enters every bound.

Theorem 5 *If $d \geq 3$ and if $f \in C^{d+4}[a,b]$, then*

$$\|e''\| \leq C(\beta+1)h^{d-1}.$$

Proof. We continue to work with $x \in (x_j, x_{j+1})$, and we express the error e in (5) as

$$e(x) = \psi(x)\widetilde{e}(x),$$

where

$$\psi(x) := (x-x_j)(x-x_{j+1}), \quad \widetilde{e}(x) := \frac{A(x)}{\widetilde{B}(x)} \quad \text{and} \quad \widetilde{B}(x) := \psi(x)B(x).$$

Now, by the Leibniz rule,

$$\begin{aligned}e''(x) &= \sum_{i=0}^{2}\binom{2}{i}\psi^{(2-i)}(x)\widetilde{e}^{(i)}(x)\\ &= 2\frac{A(x)}{\widetilde{B}(x)} + 2\psi'(x)\left(\frac{A'(x)}{\widetilde{B}(x)} - A(x)\frac{\widetilde{B}'(x)}{\widetilde{B}^2(x)}\right)\\ &\quad + \psi(x)\left(\frac{A''(x)}{\widetilde{B}(x)} - 2A'(x)\frac{\widetilde{B}'(x)}{\widetilde{B}^2(x)} + 2A(x)\frac{\widetilde{B}'^2(x)}{\widetilde{B}^3(x)} - A(x)\frac{\widetilde{B}''(x)}{\widetilde{B}^2(x)}\right).\end{aligned} \tag{27}$$

Every factor $A^{(k)}(x)$ can be bounded using Lemma 2. In the coming arguments we use the following result.

Lemma 3 *If $d \geq 1$ and if $x \in [a, b]$, then*

$$\frac{1}{|\widetilde{B}(x)|} \leq Ch^{d-1}.$$

Proof. For $x \in (x_j, x_{j+1})$, the definition of $\widetilde{B}$ reads

$$\widetilde{B}(x) = \psi(x)B(x) = \psi(x)\sum_{i=0}^{n-d}\lambda_i(x).$$

Since

$$|B(x)| \geq |\lambda_i(x)|,$$

for any $i \in J_2$, we deduce from the definition of λ_i that

$$\frac{1}{|\widetilde{B}(x)|} \leq \frac{\prod_{k=i}^{i+d} d_k}{d_j d_{j+1}} = \prod_{\substack{k=i \\ k\neq j,j+1}}^{i+d} d_k \leq Ch^{d-1}, \quad \forall i \in J_2, \tag{28}$$

which evidently holds also at the nodes. $\square$

The factors which remain to be bounded are the following:

$$N_1(x) := \frac{\widetilde{B}'(x)}{\widetilde{B}^2(x)}, \quad N_2(x) := \psi(x)\frac{\widetilde{B}'(x)}{\widetilde{B}(x)}, \quad N_3(x) := \psi(x)\frac{\widetilde{B}''(x)}{\widetilde{B}^2(x)}. \tag{29}$$

We split $\widetilde{B}$ into five parts:

$$\begin{aligned}\widetilde{B}(x) &= \psi(x)\left(\sum_{i=0}^{j-d-1}\lambda_i(x) + \lambda_{j-d}(x) + \sum_{i=j-d+1}^{j}\lambda_i(x) + \lambda_{j+1}(x) + \sum_{i=j+2}^{n-d}\lambda_i(x)\right) \\ &=: K_1(x) + K_2(x) + K_3(x) + K_4(x) + K_5(x).\end{aligned} \tag{30}$$

For symmetry reasons, it is sufficient to study the first three terms, K_1, K_2 and K_3, since K_4 and K_5 are analogous to K_2 and K_1. We begin with the first derivative of K_1:

$$K_1'(x) = \psi'(x)\sum_{i=0}^{j-d-1}\lambda_i(x) + \psi(x)\sum_{i=0}^{j-d-1}\lambda_i'(x). \tag{31}$$

The terms in both sums alternate in sign and increase in absolute value; we deduce that

$$|K_1'(x)| \leq 2h\prod_{k=j-d-1}^{j-1} d_k^{-1} + d_j d_{j+1}\sum_{m=j-d-1}^{j-1} d_m^{-1}\prod_{k=j-d-1}^{j-1} d_k^{-1}. \tag{32}$$

We next turn to K_2, which after simplification reads

$$K_2(x) = (x - x_{j+1})(-1)^{j-d} \prod_{k=j-d}^{j-1} (x - x_k)^{-1}. \tag{33}$$

It follows that

$$|K_2'(x)| \leq \prod_{k=j-d}^{j-1} d_k^{-1} + d_{j+1} \sum_{m=j-d}^{j-1} d_m^{-1} \prod_{k=j-d}^{j-1} d_k^{-1}. \tag{34}$$

We may rewrite K_3 as

$$K_3(x) = \sum_{i=j-d+1}^{j} (-1)^i \prod_{\substack{k=i \\ k\neq j,j+1}}^{i+d} (x - x_k)^{-1}, \tag{35}$$

which yields the following bound for its derivative:

$$|K_3'(x)| \leq \sum_{i=j-d+1}^{j} \sum_{\substack{m=i \\ m\neq j,j+1}}^{i+d} d_m^{-1} \prod_{\substack{k=i \\ k\neq j,j+1}}^{i+d} d_k^{-1}. \tag{36}$$

In view of deriving a bound on N_1, we first take the quotient of (32) with $|\widetilde{B}(x)|^2$. Choosing $i = j - d + 1$ in (28), we obtain

$$\begin{aligned}
\frac{|K_1'(x)|}{|\widetilde{B}(x)|^2} &\leq 2h \frac{\prod_{k=j-d+1}^{j-1} d_k^2}{\prod_{k=j-d-1}^{j-1} d_k} + d_{j+1} \sum_{m=j-d-1}^{j-1} \frac{d_j \prod_{k=j-d+1}^{j-1} d_k^2}{d_m \prod_{k=j-d-1}^{j-1} d_k} \\
&\leq 2h \frac{\prod_{k=j-d+1}^{j-1} d_k}{d_{j-d-1} d_{j-d}} + h \sum_{m=j-d-1}^{j-1} \frac{d_j \prod_{k=j-d+1}^{j-1} d_k}{d_m d_{j-d-1} d_{j-d}}.
\end{aligned}$$

Since $d_{j-d+1}/d_{j-d-1} \leq 1$ and $d_{j-d+2}/d_{j-d} \leq 1$, and $d_j/d_m \leq 1$ for $m \leq j-1$, we see that

$$\frac{|K_1'(x)|}{|\widetilde{B}(x)|^2} \leq 2h \prod_{k=j-d+3}^{j-1} d_k + h \sum_{m=j-d-1}^{j-1} \prod_{k=j-d+3}^{j-1} d_k \leq Ch^{d-2}.$$

Using similar arguments, a bound of the same order may be derived for $|K_2'|/|\widetilde{B}|^2$ and for $|K_3'|/|\widetilde{B}|^2$. The result is

$$|N_1(x)| \leq Ch^{d-2}.$$

To deal with N_2, we use the mesh ratio β. Again we begin with the term involving K_1', choose $i = j - d + 1$ in (28) and, instead of cancelling factors in the numerator and denominator, we use the fact that $d_k/d_{k-1} \leq 1$ for $k = j - d + 1, \ldots, j - 1$:

$$\begin{aligned}
|\psi(x)| \frac{|K_1'(x)|}{|\widetilde{B}(x)|} &\leq 2h \frac{d_j d_{j+1} \prod_{k=j-d+1}^{j-1} d_k}{\prod_{k=j-d-1}^{j-1} d_k} + d_{j+1}^2 \sum_{m=j-d-1}^{j-1} \frac{d_j^2 \prod_{k=j-d+1}^{j-1} d_k}{d_m \prod_{k=j-d-1}^{j-1} d_k} \\
&\leq 2h \frac{d_j d_{j+1}}{d_{j-d-1} d_{j-1}} + h \sum_{m=j-d-1}^{j-1} \frac{d_j^2 d_{j+1}}{d_m d_{j-d-1} d_{j-1}}.
\end{aligned}$$

Since $d_j/d_{j-d-1} \le 1$, and $d_j/d_m \le 1$ for $m \le j-1$, we obtain

$$|\psi(x)|\frac{|K_1'(x)|}{|\widetilde{B}(x)|} \le 2h\frac{d_{j+1}}{d_{j-1}} + h\sum_{m=j-d-1}^{j-1}\frac{d_{j+1}}{d_{j-1}} \le C\beta h.$$

Similar arguments lead to a bound of the same order for $|\psi(x)||K_2'|/|\widetilde{B}|$. For $|\psi(x)||K_3'|/|\widetilde{B}|$ we may cancel the whole product in every term of the inner sum without making use of the mesh ratio:

$$|\psi(x)|\frac{|K_3'(x)|}{|\widetilde{B}(x)|} \le d_j d_{j+1}\sum_{i=j-d+1}^{j}\sum_{\substack{m=i\\ m\ne j,j+1}}^{i+d} d_m^{-1} \le Ch.$$

Thus we have

$$|N_2(x)| \le C(\beta+1)h.$$

A bound for N_3 may be derived using similar arguments as for N_1 and the following observation: the differentiation of $\widetilde{B}'$ leads to a supplementary factor $(x-x_i)^{-1}$ in some of the terms of $\widetilde{B}''$. Since $i \ne j, j+1$, the absolute value of this factor can be eliminated through multiplication with $|\psi|$:

$$\frac{|\psi(x)|}{|x-x_i|} = \frac{d_j d_{j+1}}{d_i} \le Ch.$$

Consequently

$$|N_3(x)| \le Ch^{d-1}.$$

This last step concludes the proof, since bringing together all the bounds on the terms of the expansion (27) of e'' yields the claimed result. □

Theorem 6 *If $d = 2$ and if $f \in C^6[a,b]$, then*

$$\|e''\| \le C(\beta^2+\beta+1)h.$$

Proof. If we expand again the factors N_1, N_2 and N_3 in (29) in the special case $d = 2$, we see that everyone of them may be bounded by a linear function of β. □

4 Numerical examples

In order to evaluate the derivatives of r, we write it in its barycentric form

$$r(x) = \frac{\sum_{i=0}^{n}\frac{w_i}{x-x_i}f_i}{\sum_{i=0}^{n}\frac{w_i}{x-x_i}}, \tag{37}$$

where the formulas for the weights w_i are given in [7]. Elegant formulas for the derivatives of an interpolant given in this form have been derived by Schneider and Werner in [11].

In particular, using their formulas for the nodal case, one can easily compute the first and second derivatives of r at all the nodes at once. Following [3], it is sufficient to construct differentiation matrices $D^{(1)}$ and $D^{(2)}$ of size $(n+1) \times (n+1)$. If $\mathbf{f}$ is the vector of length $n+1$ of the values $f(x_k)$, then the product $D^{(1)}\mathbf{f}$, respectively $D^{(2)}\mathbf{f}$, yields the vector of the first, respectively second, derivative of r at the nodes.

Table 1: Error in the derivatives of r interpolating f_1

	first derivative		second derivative	
n	error	order	error	order
10	1.2e−01		5.0e−01	
20	5.2e−03	4.5	4.5e−02	3.5
40	1.9e−04	4.7	3.3e−03	3.8
80	7.2e−06	4.7	2.5e−04	3.7
160	2.9e−07	4.6	2.1e−05	3.6
320	1.3e−08	4.5	1.9e−06	3.4
640	6.8e−10	4.3	1.9e−07	3.3

To illustrate our theoretical results, we started with an example of [7], namely the interpolation of the function $f_1(x) := \sin(x)$ for $x \in [-5, 5]$. We used the rational interpolant with $d = 4$ and sampled f_1 at equidistant nodes. Our aim was to survey the estimated approximation orders of the derivatives of the interpolant and compare them with the results obtained for the interpolant itself. We computed the error at the same eleven nodes for different values of n. Table 1 shows the errors and orders for the first and second derivatives. Comparing these results with the approximated orders in [7], we see that the order decreases almost exactly by one unit at every differentiation.

Table 2: Error in the derivatives of r interpolating f_2

	first derivative		second derivative	
n	error	order	error	order
10	4.1e−01		1.5e+00	
20	3.3e−02	3.6	2.7e−01	2.5
40	9.4e−05	8.5	1.6e−03	7.4
80	1.9e−06	5.7	7.2e−05	4.5
160	1.4e−07	3.7	1.4e−05	2.3
320	1.2e−08	3.5	2.3e−06	2.7
640	1.5e−09	3.0	3.1e−07	2.9

With the next example we studied the convergence rates at intermediate points. For that purpose, we sampled Runge's function $f_2(x) := 1/(1+x^2)$ at equidistant nodes in the interval $[-5,5]$. We chose $d = 3$ and computed the maximum error at 1000 equidistant points inside the interval which are not nodes. Table 2 displays our results, which illustrate Theorems 3 and 5 in this particular case.

Table 3: Error in the derivatives of r interpolating f_3

	first derivative		second derivative	
n	error	order	error	order
10	2.8e−01		2.0e+01	
20	7.7e−02	1.9	2.0e+00	3.3
40	1.2e−02	2.7	5.9e−01	1.7
80	1.5e−03	3.0	1.6e−01	1.9
160	2.0e−04	2.9	3.9e−02	2.0
320	2.4e−05	3.0	9.9e−03	2.0
640	3.0e−06	3.0	2.5e−03	2.0

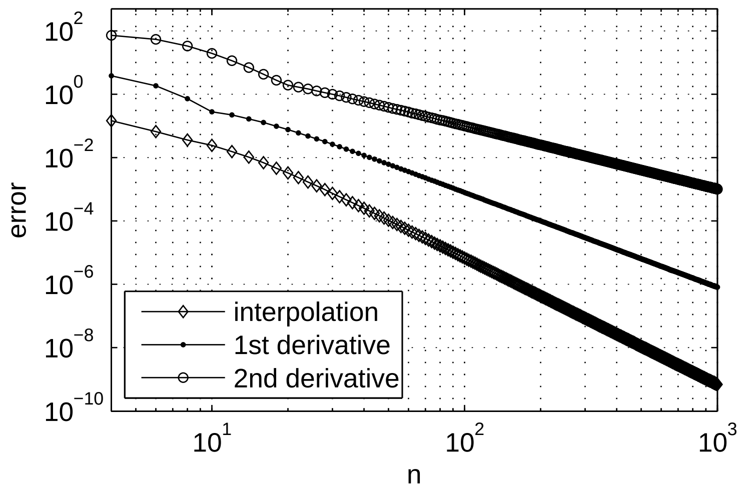

Figure 1: Errors in the rational interpolation with $d = 3$ of f_3 sampled at Chebyshev points in $[-1,1]$ and approximation of its first and second derivatives

As mentioned in the Introduction, we plan to apply the results explained in the present paper to the study of the numerical solution of differential equations. For this reason we experimented with the exact solution of a model problem from Stoer and Bulirsch [12], adapted to the interval $[-1,1]$, namely

$$f_3(x) := \frac{e^{-20}}{1+e^{-20}} e^{10(x+1)} + \frac{1}{1+e^{-20}} e^{-10(x+1)} - \cos^2\left(\frac{\pi}{2}(x+1)\right).$$

This time, we sampled the function at Chebyshev points of the second kind and interpolated the computed values using the rational interpolant with $d = 3$. Table 3 shows the maximum error at 1000 equidistant points and the experimental convergence rates. It can be proven that for such Chebyshev points of the second kind, the mesh ratio β is bounded by 3 for all n. Again, the k-th derivative of the rational interpolant converges at the rate of $O(h^{d+1-k})$ as $h \to 0$ in the cases $k = 1, 2$. We also supply a graphical survey of this same experiment at even values of n in Figure 1. In a log-log scale, the errors in the approximation of the first two derivatives of f_3 are added to those of its rational interpolant. For $n \geq 20$ the curves are nearly straight lines of slopes -4, -3 and -2.

Table 4: Error in the derivatives of the cubic spline interpolating f_2

	first derivative		second derivative	
n	error	order	error	order
10	7.6e−02		3.7e−01	
20	2.0e−02	1.9	2.9e−01	0.3
40	3.4e−03	2.5	1.1e−01	1.4
80	3.9e−04	3.1	2.5e−02	2.2
160	4.7e−05	3.0	6.1e−03	2.0
320	5.9e−06	3.0	1.5e−03	2.0
640	7.1e−07	3.1	3.8e−04	2.0

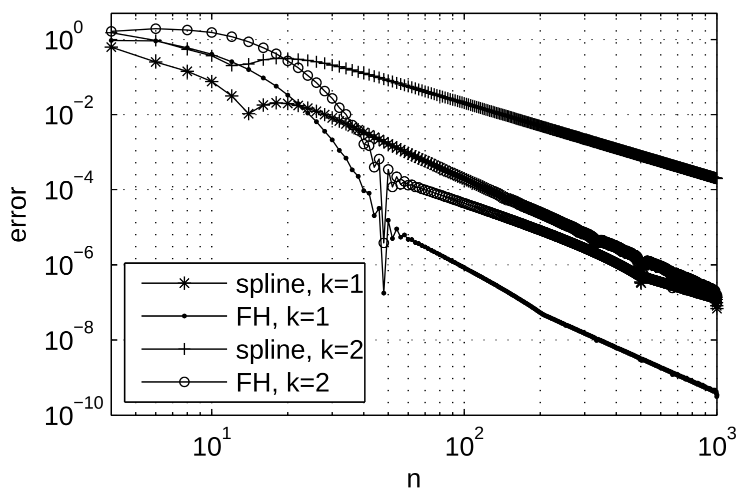

Figure 2: Errors in the spline and rational (FH) approximations with $d = 3$ of the first ($k = 1$) and second ($k = 2$) derivatives of f_2 sampled at equidistant points in $[-5, 5]$

We sampled all three functions at equidistant nodes and at Chebyshev points. The experimental convergence rates, in the cases not displayed in Tables 1, 2 and 3, are very similar and thus omitted.

Finally we repeated the computation with f_2, this time using the cubic spline with the not-a-knot end conditions. Since Runge's function is analytic, the spline interpolant and its derivatives have the same convergence orders as the rational interpolant with $d = 3$ and its derivatives (see [6]). Table 4 reveals that the experimental orders coincide for large enough n, but the error in the rational function is considerably smaller than that of the spline. The difference is due to the faster error decay of the derivatives of the rational interpolant for small values of n. Figure 2 confirms this observation: for $n \geq 50$ the curves corresponding to the errors in the spline and rational approximations of the first respectively second derivative of f_2 are almost parallel.

5 Conclusion

Our results show the $O(h^{d+1-k})$ convergence of the k-th derivative of a family of barycentric rational interpolants for $k = 1$ and 2. The question of a recursive approach for larger k's arises naturally; we tried to find such a proof, unfortunately without success in the general case studied here. However, the first and last authors have recently discovered a proof that works for the error at the nodes and under the restriction that these are equally or quasi-equally distributed [9].

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