THE HOMOLOGY OF DIGRAPHS AS A GENERALISATION OF HOCHSCHILD HOMOLOGY

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Abstract. J. Przytycki has established a connection between the Hochschild homology of an algebra A and the chromatic graph homology of a polygon graph with coefficients in A. In general the chromatic graph homology is not defined in the case where the coefficient ring is a non-commutative algebra. In this paper we define a new homology theory for directed graphs which takes coefficients in an arbitrary A–A bimodule, for A possibly non-commutative, which on polygons agrees with Hochschild homology through a range of dimensions.

1. Introduction

This paper addresses a question of J. Przytycki. When defining the Hochschild homology of an algebra with coefficients in a bimodule, the differential displays a certain cyclical feature which makes it sometimes convenient to write the tensor factors of n-chain generators, not linearly, but instead as the vertices of an $(n+1)$-sided polygon. Przytycki has a beautiful interpretation of this appearance of n-gons in terms of the chromatic homology of graphs developed by of Helme-Guizon and Rong ([3]): a variant of this theory constructed using an algebra A and an $A{-}A$-bimodule M applied to the n-gon is the Hochschild homology of A with coefficients in M through a range of dimensions increasing with n. In general chromatic homology of a graph is only defined when the algebra is commutative, but in the case of the n-gon or a line graph, the non-commutative case also makes sense. In speculating about possible generalisations of Hochschild homology in [5] Przytycki writes that he believes "graph homology is the proper generalization of Hochschild homology: from a polygon to any graph". The remaining problem being, however, that one does not know in general how to define chromatic graph homology involving non-commutative algebras. To paraphrase, Przytycki's question is:

> *Given a (possibly non-commutative) algebra A and an A–A bimodule M, can one construct a functor from some category of graphs to graded modules such that on n-gons this functor agrees with Hochschild homology through a range of dimensions ?*

In this paper we consider finite based directed graphs, that is digraphs whose vertex set is finite and for which there is a distinguished vertex, the *base vertex.* Given a triple (Γ, A, M) consisting of a based digraph Γ, an R-algebra A and an $A{-}A$ bimodule M our main purpose is to define homology groups $\mathcal{H}_*(\Gamma, A, M)$

1991 *Mathematics Subject Classification.* 16E40, 05C20, 57M15.

Key words and phrases. Chromatic graph homology, Hochschild homology, Directed graph.

The first author was partially supported by Swiss National Science Foundation projects no. 200020-121506 and no. 200021-131967/1.

with nice functorial properties. There are two key ideas necessary to embrace non-commutative algebras: firstly one must give up on having "cube" (as seen in the construction of chromatic graph homology) and secondly one needs directed edges in order to be able to multiply non-commuting elements (the head and tail providing information on which element comes first). The base vertex is necessitated by the appearance of the bimodule M. In order to obtain a functor it is essential that we use the approach initiated in [1] relating Khovanov-type homology to the homology of posets with coefficients in a presheaf.

The construction goes roughly as follows. We replace a digraph Γ by its poset of directed multipaths $P(\Gamma)$ and then define the homology groups $H_*(\Gamma, \mathcal{F})$ as the homology of the poset $P(\Gamma)$ with coefficients in an arbitrary coefficient system (presheaf) $\mathcal{F}$. Armed with these generalities we construct from a pair (A, M) where A is an algebra and M an A–A bimodule, a particular coefficient system $\mathcal{F}_{A,M}$. This coefficient system generalises the construction of the chromatic homology of graphs. The homology groups we are after are then defined by

$$\mathcal{H}_*(\Gamma, A, M) = H_*(\Gamma, \mathcal{F}_{A,M}).$$

These homology groups satisfy some nice properties and give one possible answer to Przytycki's question. Let $\mathbf{DirGr}_b$ denote the category of finite based digraphs with base vertex preserving inclusions of digraphs as morphisms and let $\mathbf{GrMod}$ denote the category of $\mathbb{Z}$-graded R-modules. We then have:

Theorem 3.5 *Let A be an R-algebra and M an A–A bimodule. Then*

$$\mathcal{H}_*(-, A, M)\colon \mathbf{DirGr}_b \to \mathbf{GrMod}$$

is a functor with the property that if Γ is a consistently directed n-gon then for $0 \leq i \leq n-2$

$$\mathcal{H}_i(\Gamma, A, M) \cong HH_i(A; M)$$

where on the right hand side we have the Hochschild homology of the algebra A with coefficients in the bimodule M.

An interesting special case arises when $M = A$. In this case unbased digraphs suffice and we write $\mathcal{H}_*(\Gamma, A) = \mathcal{H}_*(\Gamma, A, A)$. This is functorial in both variables:

Theorem 3.6 $\mathcal{H}_*(-,-)\colon \mathbf{DirGr} \times \mathbf{Alg} \to \mathbf{GrMod}$ *is a bifunctor.*

By the previous theorem this bifunctor has the property if Γ is a consistently directed n-gon then for $0 \leq i \leq n-2$,

$$\mathcal{H}_i(\Gamma, A) \cong HH_i(A).$$

We note that there is a possible alternative construction based on the fact that $P(\Gamma)$ can be viewed as a sub-poset of the Boolean lattice (cube) of all (undirected) subgraphs, but this approach fails to give the functoriality we want: for Khovanov-type complexes the inclusion of a sub-poset does not in general induce a chain map.

2. The homology of directed graphs

In this section we will be dealing with the category **DirGr** whose objects are finite directed graphs (unbased) and whose morphisms are inclusions of directed graphs. Our interest will be to construct homology of directed graphs for an arbitrary coefficient system. Note that a directed graph Γ comes with *tail* and *head* functions

$$t, h\colon Edge(\Gamma) \to Vert(\Gamma)$$

taking a directed edge e to its tail and its head respectively.

We will assume familiarity with the basics of posets. As usual we will denote a partial ordering by $\leq$ with $x < y$ meaning $x \leq y$ and $x \neq y$. We recall that an element y is said to *cover* another element x if $x < y$ and there is no z such that $x < z < y$. In such a circumstance we write $x \prec y$. The *Hasse diagram* of a poset P is the directed graph with one vertex for each element of P and an oriented arc from x to y if and only if $x \prec y$. The *Boolean lattice* on a set is the poset of subsets partially ordered by inclusion and its Hasse diagram is a hypercube. A poset may be regarded as a category with one object for each element and a unique morphism from x to y whenever $x \leq y$. Such morphisms compose in the obvious way.

Let Γ be a finite digraph and let $\mathbb{B}(\Gamma)$ be the Boolean lattice on its edge set $Edge(\Gamma)$.

Definition 2.1. A *simple path* in Γ is a sequence of edges $e_1, \ldots, e_n$ such that $h(e_i) = t(e_{i+1})$ and no vertex is encountered twice. A *multipath* in Γ is a collection of disjoint simple paths.

Let $P(\Gamma)$ be the subposet of $\mathbb{B}(\Gamma)$ consisting of multipaths in Γ. We will refer to it as the *path poset* of Γ. By convention there is one empty path $\emptyset$ and this is a (global) minimal element for $P(\Gamma)$ which we will denote by 0. There may be several (local) maxima. As mentioned in the previous paragraph we may choose to view $P(\Gamma)$ as a category with a unique morphism between any two related elements.

Example 2.2. In Figure 1 we see a digraph Γ along with an illustration of its path poset.

The assignment of a digraph to its path poset gives a covariant functor

$$P(-)\colon \mathbf{DirGr} \to \mathbf{Posets}.$$

Given an inclusion $f\colon \Gamma' \to \Gamma$ we write $\tilde{f}\colon P(\Gamma') \to P(\Gamma)$ for $P(f)$. Note that this is an injective map of posets with the property that if $x \prec y$ in $P(\Gamma')$ then $\tilde{f}(x) \prec \tilde{f}(y)$ in $P(\Gamma)$.

We note for later use the following lemma concerning such path posets.

Lemma 2.3.

(i) *For $x \in P(\Gamma)$, the interval $[0, x] = \{y \in P(\Gamma) \mid y \leq x\}$ is a Boolean lattice of rank $|x|$.*

(ii) *If $x \prec y \prec z$ in $P(\Gamma)$ then there exists a unique $y' \neq y$ such that $x \prec y' \prec z$.*

Proof. For (i) we simply note that given a multipath then each subset of its edges is again a multipath. For (ii), the stated property is true for Boolean lattices and so the result follows by considering the interval $[0, z]$. $\square$

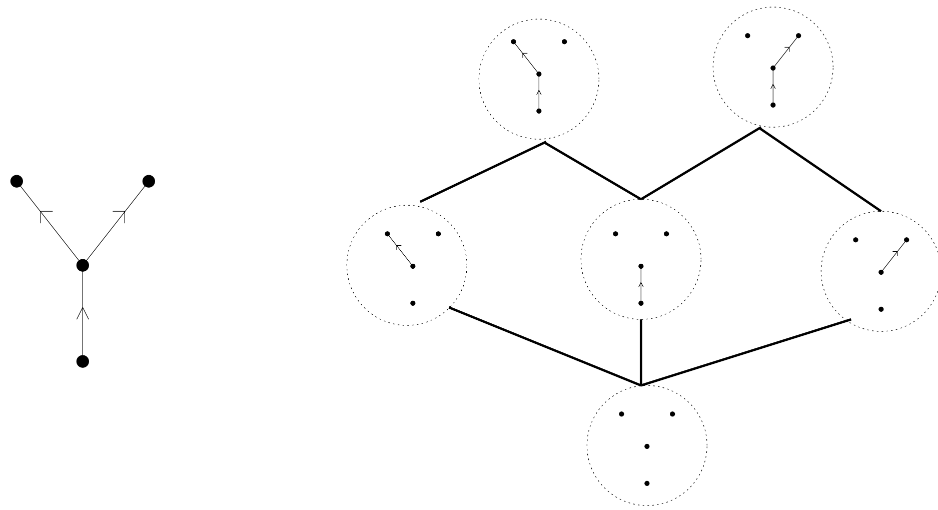

FIGURE 1. A digraph and its path poset

In general the homology of posets only becomes interesting (especially in the presence of a global minimum) if one allows local systems of coefficients. Let R be a commutative ring and let **Mod** be the category of R-modules.

Definition 2.4. A *coefficient system* for a digraph Γ consists of a covariant functor $P(\Gamma) \to \mathbf{Mod}$. These form the objects of a category $\mathrm{Coeff}(\Gamma)$ whose morphisms are natural transformations of functors.

One can take the homology of any small category with coefficients in a functor (see for example [4] section II.6) and we now recall this construction restricted to our particular setting i.e. where the category is a path category of a digraph Γ and the functor $\mathcal{F}\colon P(\Gamma) \to \mathbf{Mod}$ is a coefficient system. We will define a chain complex $\mathcal{S}_*(\Gamma, \mathcal{F})$ whose homology, by definition, is the homology of Γ with coefficients in $\mathcal{F}$.

We set

$$\mathcal{S}_k(\Gamma; \mathcal{F}) = \bigoplus_{x_0 x_1 \dots x_k} \mathcal{F}(x_0)$$

where the sum is over all sequences $x_0 \leq x_1 \leq \cdots \leq x_k$ in $P(\Gamma)$ of length $k+1$. A typical element is therefore a sum of elements of the form $\lambda x_0 x_1 \dots x_k$ where $\lambda \in \mathcal{F}(x_0)$. To turn this into a complex we define $d\colon \mathcal{S}_k(\Gamma; \mathcal{F}) \to \mathcal{S}_{k-1}(\Gamma; \mathcal{F})$ by

$$d(\lambda x_0 x_1 \dots x_k) = \mathcal{F}(x_0 \leq x_1)(\lambda) x_1 \dots x_k + \sum_{i=1}^{k} (-1)^i \lambda x_0 \dots \hat{x}_i \dots x_k.$$

It is a standard fact (easily checked) that $d^2 = 0$ and so $(\mathcal{S}_*(\Gamma; \mathcal{F}), d)$ is a chain complex. We are now free to take homology.

Definition 2.5. The *homology of the directed graph* Γ *with coefficients in* $\mathcal{F}$ is the graded R-module

$$H_*(\Gamma; \mathcal{F}) = H(\mathcal{S}_*(\Gamma; \mathcal{F}), d).$$

This homology satisfies some nice functorial properties as we see in the next proposition.

Proposition 2.6.

(i) *Let* Γ *be a finite directed graph. Then*

$$H_*(\Gamma; -) \colon \mathit{Coeff}(\Gamma) \to \mathbf{Mod}$$

is a covariant functor.

(ii) *Let* $f \colon \Gamma' \to \Gamma$ *be an inclusion and let* $\mathcal{F} \colon P(\Gamma) \to \mathbf{Mod}$ *be a coefficient system for* Γ*. Then there is an induced homomorphism*

$$f_* \colon H_*(\Gamma'; \mathcal{F} \circ \tilde{f}) \to H_*(\Gamma; \mathcal{F}).$$

Such induced homomorphisms are well behaved under composition of inclusions.

Proof. (i) Let $\mathcal{F}$ and $\mathcal{G}$ be coefficient systems and let τ be a morphism from $\mathcal{F}$ to $\mathcal{G}$. That is, $\mathcal{F}$ and $\mathcal{G}$ are functors $P(\Gamma) \to \mathbf{Mod}$ and τ is a natural transformation $\mathcal{F} \xrightarrow{\bullet} \mathcal{G}$ consisting of a map $\tau_x \colon \mathcal{F}(x) \to \mathcal{G}(x)$ for each $x \in P(\Gamma)$ satisfying the usual naturality requirements. We now define a homomorphism

$$\tau' \colon \mathcal{S}_k(\Gamma; \mathcal{F}) \to \mathcal{S}_k(\Gamma; \mathcal{G})$$

by setting

$$\tau'(\lambda x_0 \dots x_k) = \tau_{x_0}(\lambda) x_0 \dots x_k.$$

The naturality of τ guarantees that this is a chain map and thus induces

$$\tau_* \colon H_*(\Gamma; \mathcal{F}) \to H_*(\Gamma; \mathcal{G})$$

as required.

Furthermore, given another natural transformation $\sigma \colon \mathcal{G} \xrightarrow{\bullet} \mathcal{K}$ one has $(\sigma\tau)' = \sigma' \circ \tau'$ from which it follows that $(\sigma\tau)_* = \sigma_* \circ \tau_*$.

(ii) Recalling that $\tilde{f}$ is the induced map on path posets, there is a homomorphism

$$f' \colon \mathcal{S}_k(\Gamma'; \mathcal{F} \circ \tilde{f}) \to \mathcal{S}_k(\Gamma; \mathcal{F})$$

defined by

$$f'(\lambda x_0 \dots x_k) = \lambda \tilde{f}(x_0) \dots \tilde{f}(x_k).$$

This is a chain map since, by definition, $\mathcal{F} \circ \tilde{f}(x \le y) = \mathcal{F}(\tilde{f}(x) \le \tilde{f}(y))$. In homology this defines f_*.

Given $g \colon \Gamma'' \to \Gamma'$ we have $\widetilde{fg} = \tilde{f} \circ \tilde{g}$ from which it follows immediately that $(fg)_* = f_* \circ g_*$.

∎

Similar calculations to those in the above proof show that the maps f_* are natural with respect to morphisms of coefficient systems. Spelt out more clearly this means the following. Let $f \colon \Gamma' \to \Gamma$ be an inclusion of finite digraphs and let $\mathcal{F}_1, \mathcal{F}_2 \colon P(\Gamma) \to \mathbf{Mod}$ be coefficient systems. Given a natural transformation $\tau \colon \mathcal{F}_1 \xrightarrow{\bullet} \mathcal{F}_2$, define a natural transformation $\tilde{\tau}$ from $\mathcal{F}_1 \circ \tilde{f}$ to $\mathcal{F}_2 \circ \tilde{f}$ by $\tilde{\tau}_x = \tau_{\tilde{f}(x)}$. Under such circumstances the following diagram commutes.

$$\begin{tikzcd}
H_*(\Gamma'; \mathcal{F}_1 \circ \tilde{f}) \arrow[r, "f_{1*}"] \arrow[d, "\tilde{\tau}_*"'] & H_*(\Gamma; \mathcal{F}_1) \arrow[d, "\tau_*"] \\
H_*(\Gamma'; \mathcal{F}_2 \circ \tilde{f}) \arrow[r, "f_{2*}"] & H_*(\Gamma; \mathcal{F}_2)
\end{tikzcd}$$

3. The homology groups $\mathcal{H}_*(\Gamma, A, M)$

The category $\mathbf{DirGr}_b$ has as objects finite digraphs that are equipped with a preferred *base vertex.* Morphisms are inclusions that take base vertex to base vertex. There is a forgetful functor $\mathbf{DirGr}_b \to \mathbf{DirGr}$ and we can take homology by first applying this functor and then proceeding as in the previous section.

Our task in this section is to construct a coefficient system $\mathcal{F}_{A,M} \colon P(\Gamma) \to \mathbf{Mod}$, given a based digraph Γ, a (possibly non-commutative) unital R-algebra A and an A–A bimodule M. Once achieved the main definition of the paper will be

$$\mathcal{H}_*(\Gamma, A, M) = H_*(\Gamma, \mathcal{F}_{A,M}).$$

We will take the tensor product of modules over unordered sets so we recall here what this means. Let S be a finite set and suppose we have a family of R-modules indexed by S, that is for each $\alpha \in S$ we have an R-module M_α. The *unordered tensor product* of this family, denoted

$$\bigotimes_{\alpha \in S} M_\alpha$$

is formed by considering all possible orderings of the set S, taking the direct sum of the ordered tensor product for each and then identifying these via the obvious canonical isomorphisms induced from permutations.

We now proceed with the construction of $\mathcal{F}_{A,M} \colon P(\Gamma) \to \mathbf{Mod}$. For $x \in P(\Gamma)$ let Γ_x be the directed graph with the same vertex set as Γ and with edge set consisting of the edges in the multipath x (along with their directions). We will write $\pi_0(\Gamma_x)$ for the set of connected components of Γ_x. We note that if $x \prec y$ then Γ_y contains all the edges of Γ_x with one addition. Since all paths are simple this additional edge clearly joins two separate components of Γ_x. There is evidently a canonical identification of the components of Γ_x and Γ_y not involved in this fusion.

For $x \in P(\Gamma)$ consider the following family of R-modules $\{M_\alpha\}$ indexed by the set $\pi_0(\Gamma_x)$. If the component indexed by α contains the base vertex then $M_\alpha = M$ otherwise $M_\alpha = A$. Now we define $\mathcal{F}_{A,M}(x)$ to be the unordered tensor product

$$\mathcal{F}_{A,M}(x) = \bigotimes_{\alpha \in \pi_0(\Gamma_x)} M_\alpha.$$

To define the homomorphisms $\mathcal{F}_{A,M}(x \leq y)$ we first consider what happens in the case $x \prec y$. Here Γ_y consists of Γ_x with an additional edge e and as noted above two distinct components in Γ_x become one in Γ_y. The idea is to define a homomorphisms using the canonical identification between components away from those that fuse, and multiplication or the actions of A on M for those that fuse. The key point is that the order of multiplication is determined by the head and tail of e.

For temporary purposes let I be an ordering of $\pi_0(\Gamma_x)$ and J be an ordering of $\pi_0(\Gamma_y)$. With respect to these orderings suppose the two components of Γ_x that fuse are indexed by i and i' and the new fused component in Γ_y is indexed by j. We now define a homomorphism (here we use the ordered tensor product)

$$\mu\colon M_i \otimes M_{i'} \longrightarrow M_j$$

by

$$\mu(a\otimes b) = \begin{cases} ab & \text{if } i \text{ indexes the component containing } t(e) \\ ba & \text{if } i \text{ indexes the component containing } h(e) \end{cases}$$

Here the expression ab has several possible meanings: if the base vertex is not involved in the fusion of components then it means the multiplication in the algebra A; if $M_i = M$ (i.e. i indexes the component containing the base vertex) then $M_j = A$ and $M_k = M$ and ab means the right action of A on M; if $M_j = M$ (i.e. j indexes the component containing the base vertex) then $M_i = A$ and $M_k = M$ and ab means the left action of A on M. One similarly interprets ba. By combining this map with the canonical permutation identification on the remaining tensor factors this gives a homomorphism of ordered tensor products $\bigotimes_I M_i \to \bigotimes_J M_j$.

Lemma 3.1. *The above defines a homomorphism*

$$\mathcal{F}_{A,M}(x \prec y)\colon \mathcal{F}_{A,M}(x) \to \mathcal{F}_{A,M}(y).$$

Proof. Let σ be a permutation taking an ordering I to another I'. In the above construction the maps μ depend on the tensor factors corresponding to $t(e)$ and $h(e)$ not on the factors position in any ordering. It follows that there is a commutative diagram

$$\begin{tikzcd} \bigotimes_I M_i \arrow[r, "\mu"] \arrow[d, "\sigma"'] & \bigotimes_J M_j \\ \bigotimes_{I'} M_{i'} \arrow[ur, "\mu"] & \end{tikzcd}$$

and so the maps μ are compatible with the symmetric group action. □

Lemma 3.2. *If $x \prec y \prec z$ and $x \prec y' \prec z$ then*

$$\mathcal{F}_{A,M}(y \prec z) \circ \mathcal{F}_{A,M}(x \prec y) = \mathcal{F}_{A,M}(y' \prec z) \circ \mathcal{F}_{A,M}(x \prec y').$$

Proof. By Lemma 2.3 (ii) we know that y and y' are the only two elements lying between x and z in this way. Suppose $\Gamma_y = \Gamma_x \cup \{e\}$ and $\Gamma_{y'} = \Gamma_x \cup \{e'\}$. If e and e' are not both contained in a single simple path of Γ_z then the result is clear. If they are both contained in the same simple path of Γ_z then without loss of generality we may suppose that e comes before e'. Now choose an ordering on the components of Γ_x so that the simple path before e is labelled 1, the simple path between e and e' is labelled 2 and the simple path after e' is labelled 3 (see Figure 2).

Now suppose we have ordering of the components of Γ_y such that 1 indexes the component of Γ_y containing e and 2 indexes the component at the head of e'. Similarly, suppose we have ordering of the components of $\Gamma_{y'}$ such that 1 indexes the component at the tail of e and 2 indexes the component containing e'. In Γ_z the simple path containing e and e' is indexed by 1. Then by the associativity of μ

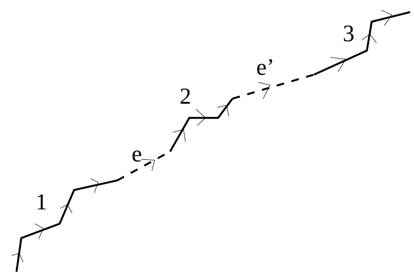

FIGURE 2. Path ordering

the following diagram commutes

$$\begin{tikzcd} M_1 \otimes M_2 \otimes M_3 \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"] & M_1 \otimes M_2 \arrow[d, "\mu"] \\ M_1 \otimes M_2 \arrow[r, "\mu"] & M_1 \end{tikzcd}$$

from which the result easily follows.

□

We now extend this to define a map $\mathcal{F}_{A,M}(x \leq y)\colon \mathcal{F}_{A,M}(x) \to \mathcal{F}_{A,M}(y)$ for any $x \leq y$. Pick a sequence $x \prec x_1 \prec \cdots \prec x_l \prec y$ and set

$$\mathcal{F}_{A,M}(x \leq y) = \mathcal{F}_{A,M}(x_l \prec y) \circ \cdots \circ \mathcal{F}_{A,M}(x \prec x_1).$$

Courtesy of Lemma 3.2 we immediately see that this does not depend on the particular choice of sequence. We have thus shown that

Proposition 3.3. $\mathcal{F}_{A,M}\colon P(\Gamma) \to \mathbf{Mod}$ *as defined above is a covariant functor, i.e.* $\mathcal{F}_{A,M}$ *is a coefficient system for* Γ.

We finally arrive at the principal definition of this section.

Definition 3.4. Let A be a unital R-algebra, M an A–A bimodule and Γ a finite based digraph. Using the coefficient system $\mathcal{F}_{A,M}$ above we define

$$\mathcal{H}_*(\Gamma, A, M) = H_*(\Gamma, \mathcal{F}_{A,M}).$$

The following theorem provides one possible answer to Przytycki's question.

Theorem 3.5. *Let* A *be an* R*-algebra and* M *an* A–A *bimodule. Then*

$$\mathcal{H}_*(-, A, M)\colon \mathbf{DirGr}_b \to \mathbf{GrMod}$$

is a functor with the property that if Γ *is a consistently directed* n*-gon then for* $0 \leq i \leq n-2$

$$\mathcal{H}_i(\Gamma, A, M) \cong HH_i(A; M)$$

where on the right hand side we have the Hochschild homology of the algebra A *with coefficients in the bimodule* M*.*

Before proving this Theorem let us recall Przytycki's result relating the chromatic homology of graphs to Hochschild homology [5]. We will state the results using homological grading conventions. Firstly recall that (homologically graded) chromatic homology of a graph G is defined as follows. Let A be a *commutative* R-algebra. Let $\mathbb{B}$ be the Boolean lattice (the "cube" as it is usually referred to) on the edges of G. An element of $\mathbb{B}$ is a subgraph of G with the same vertex set as V and will typically contain some isolated vertices. To each such subgraph x associate the module $\mathcal{F}(x)$ being a tensor product of copies of A, one for each connected component. To each edge ζ of the cube (covering relation in the Boolean lattice) associate a map d_ζ being given by the algebra multiplication if two connected components fuse, or the identity map otherwise. Letting N be the number of edges in G, one defines, for $i = 0, 1, \dots, N$

$$\mathcal{C}_i(\Gamma) = \bigoplus_{x \in P(\Gamma), |x| = N-i} \mathcal{F}(x).$$

A differential $d\colon \mathcal{C}_i(\Gamma) \to \mathcal{C}_{i-1}(\Gamma)$ can be defined for $a \in \mathcal{F}(x)$ by

$$d(a) = \bigoplus_{\zeta} \epsilon(\zeta) d_\zeta(a),$$

where $\epsilon(\zeta) = \pm 1$. This gives a complex, whose homology is the chromatic graph homology of G using the algebra A. All this is well documented elsewhere (see [2,3] for details).

Przytycki extends the above in the following way (see [5] for details). Suppose M is an A–A-bimodule (where as above A is commutative) and suppose v_1 is a chosen base vertex of G. Modify the above construction by replacing A by M whenever associating a module to a component containing v_1. Moreover, in the definition of the differential, partial derivatives between subgraphs having the same number of components are set to zero. Denote the result $\hat{H}_*^{A,M}(G)$.

If we take $G = P_n$, an n-sided polygon, then taking coherent directions on each edge (so that the whole polygon is oriented clockwise or anti-clockwise) then Przytycki argues the above may be extended to the case where A is non-commutative. (From our point of view, the existence of this global orientation, means that each subgraph is again consistently directed and is thus a multipath in our set up).

Przytycki's main result (with homological grading conventions) is:

Theorem(Przytycki)

$$\hat{H}_i^{A,M}(P_n) \cong HH_{i-1}(A; M) \qquad \text{for } 1 \leq i \leq n-1.$$

Thus, modified chromatic homology of polygons agrees with Hochschild homology through a range of dimensions. We are now ready to prove Theorem 3.5 above.

Proof. (of Theorem 3.5) Let $f\colon \Gamma' \to \Gamma$ be a (basepoint preserving) inclusion. We wish to define a chain map

$$f'\colon \mathcal{S}_*(\Gamma', \mathcal{F}'_{A,M}) \to \mathcal{S}_*(\Gamma, \mathcal{F}_{A,M})$$

Here we are writing $\mathcal{F}'_{A,M}$ for the coefficient system constructed above for the graph Γ'.

Firstly, we construct a morphism of coefficient systems $\tau\colon \mathcal{F}'_{A,M} \xrightarrow{\bullet} \mathcal{F}_{A,M} \circ \tilde{f}$. For $x \in P(\Gamma)$ the graph $\Gamma_{\tilde{f}(x)}$ is isomorphic to the graph $\Gamma'_x \cup W$ where W consists of the vertices in Γ which are not in (the image of) Γ'. We can thus make the identification

$$\mathcal{F}_{A,M}(\tilde{f}(x)) \cong \mathcal{F}'_{A,M}(x) \otimes \bigotimes_W A.$$

Moreover, if $x \prec y$ in Γ' then these identifications make the following diagram commute.

$$\begin{tikzcd}
\mathcal{F}_{A,M}(\tilde{f}(x)) \arrow[d, "\mathcal{F}_{A,M}\circ\tilde{f}(x\prec y)"'] & \cong & \mathcal{F}'_{A,M}(x)\otimes\bigotimes A \arrow[d, "\mathcal{F}'_{A,M}(x\prec y)"] \\
\mathcal{F}_{A,M}(\tilde{f}(y)) & \cong & \mathcal{F}'_{A,M}(y)\otimes\bigotimes A
\end{tikzcd}$$

We now define τ_x to be the composition

$$\mathcal{F}'_{A,M}(x) \cong \mathcal{F}'_{A,M}(x)\otimes_R \bigotimes R \to \mathcal{F}'_{A,M}(x)\otimes_R \bigotimes A \cong \mathcal{F}_{A,M}(\tilde{f}(x))$$

where the map shown is just the identity on $\mathcal{F}'_{A,M}(x)$ and the unit map of the algebra $R \to A$ on the remaining tensor factors.

For $x \prec y$ in Γ', the diagram above shows that

$$\begin{tikzcd}
\mathcal{F}'_{A,M}(x) \arrow[r, "\tau_x"] \arrow[d, "\mathcal{F}'_{A,M}(x\prec y)"'] & \mathcal{F}'_{A,M}(x)\otimes\bigotimes A \arrow[d, "\mathcal{F}'_{A,M}(x\prec y)\otimes Id"] \\
\mathcal{F}'_{A,M}(y) \arrow[r, "\tau_y"'] & \mathcal{F}'_{A,M}(y)\otimes\bigotimes A
\end{tikzcd}$$

Now suppose $x \leq y$ in Γ'. We may choose a sequence $x \prec x_1 \prec \cdots \prec x_m \prec y$ and noting that $\tilde{f}(x) \prec \tilde{f}(x_1) \prec \cdots \prec \tilde{f}(x_m) \prec \tilde{f}(y)$ we see τ is natural by repeated use of the above diagram. Thus it is thus a morphism of coefficient systems as desired.

From Proposition 2.6 (i) we now get a homomorphism

$$\tau_*\colon H_*(\Gamma', \mathcal{F}'_{A,M}) \to H_*(\Gamma', \mathcal{F}_{A,M}\circ\tilde{f}).$$

Invoking part (ii) of Proposition 2.6 we also have a homomorphism

$$f_*\colon H_*(\Gamma', \mathcal{F}_{A,M}\circ\tilde{f}) \to H_*(\Gamma, \mathcal{F}_{A,M}).$$

The composition $f_*\circ\tau_*$ gives a homomorphism

$$f_\bullet\colon \mathcal{H}_*(\Gamma', A, M) \to \mathcal{H}_*(\Gamma, A, M)$$

which is the map in homology induced by f.

It remains to show that if $g\colon \Gamma'' \to \Gamma'$ is an inclusion then $(fg)_\bullet = f_\bullet \circ g_\bullet$. This amounts to showing that the top and bottom routes around the following diagram are the same (where we have omitted the subscripts A, M and are being a little liberal in our multiple uses of the letter τ).

$$\begin{tikzcd}
& H_*(\Gamma'', \mathcal{F}'\circ\tilde{g}) \arrow[rr, "g_*"] \arrow[ddrr, "\tilde{\tau}_*"] & & H_*(\Gamma', \mathcal{F}') \arrow[rr, "\tau_*"] & & H_*(\Gamma', \mathcal{F}\circ\tilde{f}) \arrow[dr, "f_*"] & \\
H_*(\Gamma'', \mathcal{F}'') \arrow[ur, "\tau_*"] \arrow[drrr, "\tau_*"'] & & & & & & H_*(\Gamma, \mathcal{F}) \\
& & & H_*(\Gamma'', \mathcal{F}\circ\widetilde{fg}) \arrow[uurr, "g_*"] \arrow[urrr, "(fg)_*"'] & & &
\end{tikzcd}$$

The left-hand triangle commutes directly from the definition of the maps τ, and the right-hand triangle commutes from the statement in Proposition 2.6 (ii) that the induced maps behave well under composition. The middle square commutes by the comments immediately after the proof of Proposition 2.6.

In order to make the connection with Hochschild homology we combine Przytycki's result with the work of Everitt and the first author [1]. It is clear that if $\Gamma = P_n$ with consistent directions, then $P(\Gamma)$ is the Boolean lattice on the edges of P_n minus its maximum element.

Moreover the functor $\mathcal{F}_{A,M}$ constructed on $P(\Gamma)$ agrees with the (implicit) functor used in the construction of $\hat{H}^{A,M}_{i+1}$ in this case. Since the graph is a polygon, the only partial derivative in Przytycki's set-up that are set to zero are those from corank 1 elements to the maximal element and this corresponds to the absence of the maximum element of the Boolean lattice in $P(\Gamma)$.

It now follows from the main result of [1] that the homology of the path category of category P_n with coefficients in $\mathcal{F}_{A,M}$ is isomorphic to $\hat{H}^{A,M}_{i+1}(P_n)$ with a grading shift:

$$\mathcal{H}_i(P_n, A, M) \cong H_i(P_n, \mathcal{F}_{A,m}) \cong \hat{H}^{A,M}_{i+1}(P_n).$$

Combining this with Przytycki's result gives the desired isomorphism. □

When $M = A$ the base vertex become irrelevant and we may define for an (unbased) digraph Γ the homology groups $\mathcal{H}_*(\Gamma, A) = \mathcal{H}_*(\Gamma, A, A)$, where on the right-hand side any base vertex for Γ will do. Letting **Alg** denote the category of R-algebras we have:

Theorem 3.6. $\mathcal{H}_*(-,-)\colon \mathbf{DirGr} \times \mathbf{Alg} \to \mathbf{GrMod}$ *is a bifunctor.*

Proof. Functoriality in the first variable follows from the previous theorem. For the second variable, let $f\colon A \to B$ be an algebra homomorphism. For $x \in P(\Gamma)$ we have

$$\mathcal{F}_{A,A}(x) = \bigotimes A \quad \text{and} \quad \mathcal{F}_{B,B}(x) = \bigotimes B$$

where the tensor product is over the same indexing set in both cases. We can therefore define a homomorphism $\sigma_x\colon \mathcal{F}_{A,A}(x) \to \mathcal{F}_{B,B}(x)$ by $\sigma_x = \bigotimes f$. Since f is a homomorphism of algebras, the σ_x define a natural transformation $\sigma\colon \mathcal{F}_{A,A} \xrightarrow{\bullet} \mathcal{F}_{B,B}$. By Proposition 2.6 (i) this induces a map $H_*(\Gamma, \mathcal{F}_{A,A}) \to H_*(\Gamma, \mathcal{F}_{B,B})$ as required. Composition of algebra homomorphisms is easily seen to give a well defined composition of these induced maps. □

If one fixes the directed graph Γ the above gives a functor

$$\mathcal{H}^{\Gamma}_*(-)\colon \mathbf{Alg} \to \mathbf{GrMod}.$$

One is tempted to call $\mathcal{H}^{\Gamma}_*(A)$ the *homology of the algebra A with coefficients in the digraph* Γ. Such homology theories of algebras are probably worthy of study in their own right. We limit ourselves here to the observation that if γ is an oriented cycle of length n in Γ then there is an inclusion of digraphs $\gamma \to \Gamma$ which by functoriality and Theorem 3.5 gives a map

$$\gamma_*\colon HH_i(A) \to \mathcal{H}^{\Gamma}_*(A)$$

for $i = 0, 1, \ldots, n-2$.

References

[1] Brent Everitt and Paul Turner, *Homology of coloured posets: a generalization of Khovanov's cube construction*, J. Alg. **322** (2009), 429–448.

[2] Laure Helme-Guizon, Jozef Przytycki, and Yongwu Rong, *Torsion in graph homology*, Fund. Math. **190** (2006), 139–177.

[3] Laure Helme-Guizon and Yongwu Rong, *A categorification for the chromatic polynomial*, Algebr. Geom. Topol. **5** (2005), 1365–1388.

[4] I. Moerdijk, *Classifying spaces and classifying topoi*, Lecture Notes in Mathematics, vol. 1616, Springer-Verlag, Berlin, 1995.

[5] Jozef Przytycki, *When the theories meet: Khovanov homology as Hochschild homology of links*, Quantum Topology **1** (2010), no. 2, 93–99.

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