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# Classifying $k$-edge colouring for $H$-free graphs ☆

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ABSTRACT

A graph is $H$-free if it does not contain an induced subgraph isomorphic to $H$. For every integer $k$ and every graph $H$, we determine the computational complexity of $k$-Edge Colouring for $H$-free graphs.

## 1. Introduction

A graph $G = (V, E)$ is *$k$-edge colourable* for some integer $k$ if there exists a mapping $c : E \to \{1, \ldots, k\}$ such that $c(e) \neq c(f)$ for any two edges $e$ and $f$ of $G$ that have a common end-vertex. The *chromatic index* of $G$ is the smallest integer $k$ such that $G$ is $k$-edge colourable. Vizing proved the following classical result.

**Theorem 1** *([27]). The chromatic index of a graph $G$ with maximum degree $\Delta$ is either $\Delta$ or $\Delta + 1$.*

The Edge Colouring problem is to decide if a given graph $G$ is $k$-edge colourable for some given integer $k$. Note that $(G, k)$ is a yes-instance if $G$ has maximum degree at most $k-1$ by Theorem 1 and that $(G, k)$ is a no-instance if $G$ has maximum degree at least $k+1$. If $k$ is *fixed*, that is, $k$ is not part of the input, then we denote the problem by $k$-Edge Colouring. It is trivial to solve this problem for $k = 2$. However, the problem is NP-complete if $k \geq 3$, as shown by Holyer for $k = 3$ and by Leven and Galil for $k \geq 4$.

**Theorem 2** *([14,20]). For $k \geq 3$, $k$-Edge Colouring is NP-complete even for $k$-regular graphs.*

Due to the above hardness results we may wish to restrict the input to some special graph class. A natural property of a graph class is to be closed under vertex deletion. Such graph classes are called *hereditary* and they form the focus of our paper. To give an example, bipartite graphs form a hereditary graph class, and it is well-known that they have chromatic index $\Delta$. Hence, Edge Colouring is polynomial-time solvable for bipartite graphs, which are perfect and triangle-free. In contrast, Cai and Ellis [4] proved that for every $k \geq 3$, $k$-Edge Colouring is NP-complete for $k$-regular comparability graphs, which are also perfect. They also proved the following two results, the first one of which shows that Edge Colouring is NP-complete for triangle-free graphs (the graph $C_s$ denotes the cycle on $s$ vertices).

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**Theorem 3** *([4]). Let $k \geq 3$ and $s \geq 3$. Then $k$-Edge Colouring is NP-complete for $k$-regular $C_s$-free graphs.*

**Theorem 4** *([4]). Let $k \geq 3$ be an odd integer. Then $k$-Edge Colouring is NP-complete for $k$-regular line graphs of bipartite graphs.*

It is also known that Edge Colouring is polynomial-time solvable for chordless graphs [22], series-parallel graphs [16], split-indifference graphs [26] and for graphs of treewidth at most $k$ for any constant $k$ [1].

It is not difficult to see that a graph class $\mathcal{G}$ is hereditary if and only if it can be characterized by a set $\mathcal{F}_{\mathcal{G}}$ of forbidden induced subgraphs (see, for example, [17]). Malyshev determined the complexity of 3-Edge Colouring for every hereditary graph class $\mathcal{G}$, for which $\mathcal{F}_{\mathcal{G}}$ consists of graphs that each have at most five vertices, except perhaps two graphs that may contain six vertices [23]. Malyshev performed a similar complexity study for Edge Colouring for graph classes defined by a family of forbidden (but not necessarily induced) graphs with at most seven vertices and at most six edges [24].

We focus on the case where $\mathcal{F}_{\mathcal{G}}$ consists of a single graph $H$. A graph $G$ is *$H$-free* if $G$ does not contain an induced subgraph isomorphic to $H$. We obtain the following dichotomy for $H$-free graphs.

**Theorem 5.** *Let $k \geq 3$ be an integer and $H$ be a graph. If $H$ is a linear forest, then $k$-Edge Colouring is polynomial-time solvable for $H$-free graphs. Otherwise $k$-Edge Colouring is NP-complete even for $k$-regular $H$-free graphs.*

We obtain Theorem 5 by combining Theorems 3 and 4 with two new results. In particular, we will prove a hardness result for $k$-regular claw-free graphs for even integers $k$ (as Theorem 4 is only valid when $k$ is odd).

## 2. Preliminaries

The graphs $C_n$, $P_n$ and $K_n$ denote the path, cycle and complete graph on $n$ vertices, respectively. A set $I$ is an *independent set* of a graph $G$ if all vertices of $I$ are pairwise nonadjacent in $G$. A graph $G$ is *bipartite* if its vertex set can be partitioned into two independent sets $A$ and $B$. If there exists an edge between every vertex of $A$ and every vertex of $B$, then $G$ is *complete bipartite*. The *claw* $K_{1,3}$ is the complete bipartite graph with $|A| = 1$ and $|B| = 3$.

Let $G_1$ and $G_2$ be two vertex-disjoint graphs. The *join* operation $\times$ adds an edge between every vertex of $G_1$ and every vertex of $G_2$. The *disjoint union* operation $+$ merges $G_1$ and $G_2$ into one graph without adding any new edges, that is, $G_1 + G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. We write $rG$ to denote the disjoint union of $r$ copies of a graph $G$.

A *forest* is a graph with no cycles. A *linear forest* is a forest of maximum degree at most 2, or equivalently, a disjoint union of one or more paths. A graph $G$ is a *cograph* if $G$ can be generated from $K_1$ by a sequence of join and disjoint union operations. A graph is a cograph if and only if it is $P_4$-free (see, for example, [3]). The following well-known lemma follows from this equivalence and the definition of a cograph.

**Lemma 1.** *Every connected $P_4$-free graph on at least two vertices has a spanning complete bipartite subgraph.*

Let $G = (V, E)$ be a graph. For a subset $S \subseteq V$, the graph $G[S] = (S, \{uv \in E \mid u, v \in S\})$ denotes the subgraph of $G$ *induced* by $S$. We say that $S$ is *connected* if $G[S]$ is connected. Recall that a graph $G$ is *$H$-free* for some graph $H$ if $G$ does not contain $H$ as an induced subgraph. A subset $D \subset V(G)$ is *dominating* if every vertex of $V(G) \setminus D$ is adjacent to least one vertex of $D$. We will need the following result of Camby and Schaudt.

**Theorem 6** *([5]). Let $t \geq 4$ and $G$ be a connected $P_t$-free graph. Let $X$ be any minimum connected dominating set of $G$. Then $G[X]$ is either $P_{t-2}$-free or isomorphic to $P_{t-2}$.*

Let $G = (V, E)$ be some graph. The *degree* of a vertex $u \in V$ is equal to the size of its neighbourhood $N(u) = \{v \mid uv \in E\}$. The graph $G$ is *$r$-regular* if every vertex of $G$ has degree $r$. The *line graph* of $G$ is the graph $L(G)$, which has vertex set $E$ and an edge between two distinct vertices $e$ and $f$ if and only if $e$ and $f$ have a common end-vertex in $G$.

## 3. The Proof of Theorem 5

To prove our dichtomy, we first consider the case where the forbidden induced subgraph $H$ is a claw. As line graphs are claw-free, we only need to deal with the case where the number of colours $k$ is even due to Theorem 4. For proving this case we need another result of Cai and Ellis, which we will use as a lemma. Let $c$ be a $k$-edge colouring of a graph $G = (V, E)$. Then a vertex $u \in V$ *misses* colour $i$ if none of the edges incident to $u$ is coloured $i$.

**Lemma 2** *([4]). For even $k \geq 2$, the complete graph $K_k$ has a $k$-edge colouring with the property that $V(K_k)$ can be partitioned into sets $\{u_i, u_i'\}$ $(1 \leq i \leq \frac{k}{2})$, such that for $i = 1, \ldots, \frac{k}{2}$, vertices $u_i$ and $u_i'$ miss the same colour, which is not missed by any of the other vertices.*

We use Lemma 2 to prove the following result, which solves the case where $k$ is even and $H = K_{1,3}$.

**Lemma 3.** *Let $k \geq 4$ be an even integer. Then $k$-Edge Colouring is NP-complete for $k$-regular claw-free graphs.*

**Proof.** Recall that $k$-Edge Colouring for $k$-regular graphs is NP-complete for every integer $k \geq 4$ due to Theorem 2. Consider an instance $(G, k)$ of $k$-Edge Colouring, where $G$ is $k$-regular for some even integer $k = 2\ell \geq 4$. From $G$ we construct a graph $G'$ as follows. First we replace every vertex $v$ in $G$ by the gadget $H(v)$ shown in Fig. 1. Next we connect the different gadgets in the following way. Every gadget $H(v)$ has exactly $k$ pendant edges, which are incident with vertices $v_1, \ldots, v_\ell, v_{\ell+1}, \ldots, v_{2\ell}$, respectively. As $G$ is $k$-regular, every vertex has $k$ neighbours in $G$.

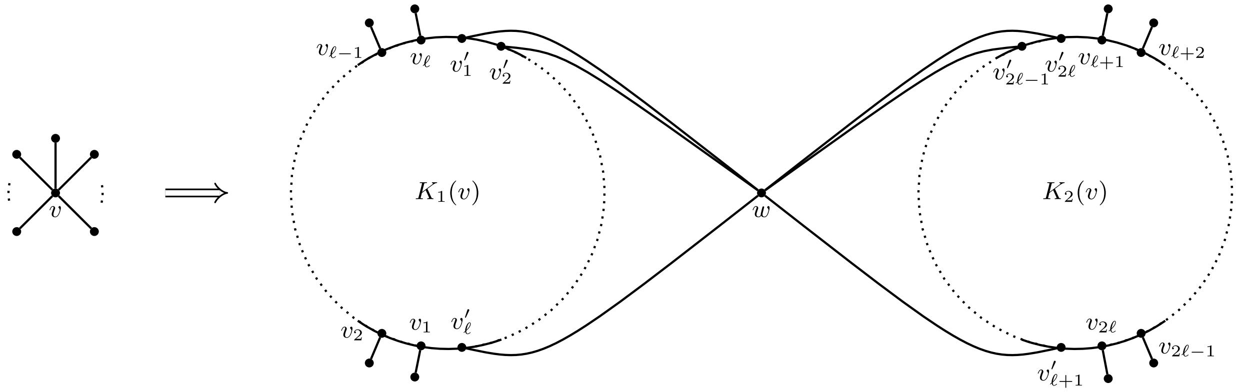


**Fig. 1.** The gadget $H(v)$ where $K_i(v)$ is a complete graph of size $2\ell$ for $i = 1, 2$. Note that edges inside $K_1(v)$ and $K_2(v)$ are not drawn.

Hence, we can identify each edge $uv$ of $G$ with a unique edge $u_h v_i$ in $G'$, which is a pendant edge of both $H(u)$ and $H(v)$. It is readily seen that $G'$ is $k$-regular and claw-free.

First suppose that $G$ is $k$-edge colourable. Let $c$ be a $k$-edge colouring of $G$. Consider a vertex $v \in V(G)$. For every neighbour $u$ of $v$ in $G$, we colour the pendant edge in $H(v)$ corresponding to the edge $uv$ with colour $c(uv)$. As $c$ assigned different colours to the edges incident to $v$, the $2\ell$ pendant edges of $H(v)$ will receive pairwise distinct colours, which we denote by $x_1, \dots, x_\ell, y_1, \dots, y_\ell$. By Lemma 2, we can colour the edges of $K_1(v)$ in such a way that for $i = 1, \dots, \ell$, $v_i$ and $v'_i$ miss colour $x_i$. For $i = 1, \dots, \ell$, we can therefore assign colour $x_i$ to edge $v'_i w$. Similarly, we may assume that for $i = 1, \dots, \ell$, $v_{\ell+i}$ and $v'_{\ell+i}$ miss colour $y_i$. For $i = 1, \dots, \ell$, we can therefore assign colour $y_i$ to edge $v'_{\ell+i} w$. Recall that the colours $x_1, \dots, x_\ell$, $y_1, \dots, y_\ell$ are all different. Hence, doing this procedure for each vertex of $G$ yields a $k$-edge colouring $c'$ of $G'$.

Now suppose that $G'$ is $k$-edge colourable. Let $c'$ be a $k$-edge colouring of $G'$. Consider some $v \in V(G)$. Denote the pendant edges of $H(v)$ by $e_i$ for $i = 1, \dots, 2\ell$, where $e_i$ is incident to $v_i$ (and to some vertex $u_h$ in a gadget $H(u)$ for each neighbour $u$ of $v$ in $G$). Suppose that $c'$ gave colour $x$ to an edge $wv'_i$ for some $1 \le i \le \ell$, say to $wv'_1$, but not to any edge $e_i$ for $i = 1, \dots, \ell$. Note that $wv'_2, \dots, wv'_\ell$ cannot be coloured $x$. As every vertex of $G'$ has degree $k = 2\ell$, every $v_i$ with $1 \le i \le \ell$ and every $v'_j$ with $2 \le j \le \ell$ is incident to some edge coloured $x$. As $x$ is neither the colour of $e_1, \dots, e_\ell$ nor the colour of $wv'_2, \dots, wv'_\ell$, the complete graph $K_1(v) - v'_1$ contains a perfect matching all of whose edges have colour $x$. However, $K_1(v) - v'_1$ has odd size $2\ell - 1$. Hence, this is not possible. We conclude that each of the (pairwise distinct) colours of $wv'_1, \dots, wv'_\ell$, which we denote by $x_1, \dots, x_\ell$, is the colour of an edge $e_i$ for some $1 \le i \le \ell$.

Let $y_1, \dots, y_\ell$ be the (pairwise distinct) colours of $wv'_{\ell+1}, \dots, wv'_{2\ell}$, respectively. By the same arguments as above, we find that each of those colours is also the colour of a pendant edge of $H(v)$ that is incident to a vertex $v_{\ell+i}$ for some $1 \le i \le \ell$. Note that $x_1, \dots, x_\ell$, $y_1, \dots, y_\ell$ are $2\ell$ pairwise distinct colours, as they are colours of edges incident to the same vertex, namely vertex $w$. Hence, we can define a $k$-colouring $c$ of $G$ by setting $c(uv) = c'(u_h v_i)$ for every edge $uv \in E(G)$ with corresponding edge $u_h v_i \in E(G')$. $\square$

We note that the graph $G'$ in the proof of Lemma 3 is not a line graph, as the gadget $H(v)$ is not a line graph: the vertices $v'_1, v'_2, v_1, w$ form a diamond and by adding the pendant edge incident to $v_1$ and the edge $wv'_{\ell+1}$ we obtain an induced subgraph of $H(v)$ that is not a line graph.

To handle the case where the forbidden induced subgraph $H$ is a path, we make the following observation.

**Observation 1.** *If a graph $G$ of maximum degree $k$ has a dominating set of size at most $p$, then $G$ has at most $p(k+1)$ vertices.*

We use Observation 1 to prove the following lemma.

**Lemma 4.** *Let $k \ge 0$ and $t \ge 1$. Every connected $P_t$-free graph of maximum degree $k$ has at most $f(k,t)$ vertices for some function $f$ that only depends on $k$ and $t$.*

**Proof.** Let $G$ be a connected $P_t$-free graph of maximum degree at most $k$. We use induction on $t$.

First suppose $t = 4$ (and observe that if the claim holds for $t = 4$, it also holds for $t \le 3$). As $G$ is connected, $G$ has a dominating set of size 2 due to Lemma 1. Hence, by Observation 1, we find that $G$ has at most $f(k, 2) = 2(k+1)$ vertices.

Now suppose $t \ge 5$. Let $X$ be an arbitrary minimum connected dominating set of $G$. By Theorem 6, $G[X]$ is either $P_{t-2}$-free or isomorphic to $P_{t-2}$. In the first case we use the induction hypothesis to conclude that $G[X]$ has at most $f(k, t-2)$ vertices. Hence, $G$ has at most $f(k, t-2)(k+1)$ vertices by Observation 1. In the second case, we find that $G$ has at most $(t-2)(k+1)$ vertices. We set $f(k,t) = \max\{f(k, t-2)(k+1), (t-2)(k+1)\}$. $\square$

We use Lemma 4 to prove our next lemma.

**Lemma 5.** *Let $k \ge 3$ and $t \ge 1$. Then $k$-Edge Colouring is linear-time solvable for $P_t$-free graphs.*

**Proof.** Let $G$ be a $P_t$-free graph. We compute the set of connected components of $G$ in linear time. For each connected component $D$ of $G$ we do as follows. We first compute in linear time the maximum degree $\Delta_D$ of $D$. If $\Delta_D \leq k-1$, then $D$ is $k$-edge colourable by Theorem 1. If $\Delta_D \geq k+1$, then $D$ is not $k$-edge colourable. Hence, we may assume that $\Delta_D = k$. By Lemma 4, $D$ has at most $f(k,t)$ vertices for some function $f$ that only depends on $k$ and $t$. As we assume that $k$ and $t$ are constants, this means that we can now check in constant time if $D$ is $k$-edge colourable. Note that $G$ is $k$-edge colourable if and only if every connected component of $G$ is $k$-edge colourable. Hence, by using the above procedure, deciding if $G$ is $k$-edge colourable takes linear time. □

We are now ready to prove Theorem 5, which we restate below.

**Theorem 5. (restated)** *Let $k \geq 3$ be an integer and $H$ be a graph. If $H$ is a linear forest, then $k$-Edge Colouring is linear-time solvable for $H$-free graphs. Otherwise $k$-Edge Colouring is NP-complete even for $k$-regular $H$-free graphs.*

**Proof.** First suppose that $H$ contains a cycle $C_s$ for some $s \geq 3$. Then the class of $H$-free graphs is a superclass of the class of $C_s$-free graphs. This means that we can apply Theorem 3. From now on assume that $H$ contains no cycle, so $H$ is a forest. Suppose that $H$ contains a vertex of degree at least 3. Then the class of $H$-free graphs is a superclass of the class of $K_{1,3}$-free graphs, which in turn forms a superclass of the class of line graphs. Hence, if $k$ is odd, then we apply Theorem 4, and if $k$ is even, then we apply Lemma 3. From now on assume that $H$ contains no cycle and no vertex of degree at least 3. Then $H$ is a linear forest, say with $\ell$ connected components. Let $t = \ell|V(H)|$. Then the class of $H$-free graphs is contained in the class of $P_t$-free graphs. Hence we may apply Lemma 5. This completes the proof of Theorem 5. □

## 4. Conclusions

We gave a complete complexity classification of $k$-Edge Colouring for $H$-free graphs, showing a dichotomy between linear-time solvable cases and NP-complete cases. We saw that this depends on $H$ being a linear forest or not. It would be interesting to prove a dichotomy result for Edge Colouring restricted to $H$-free graphs. Note that due to Theorem 5 we only need to consider the case where $H$ is a linear forest. However, even determining the complexity for small linear forests $H$, such as the cases where $H = 2P_2$ and $H = P_4$, turns out to be a difficult problem. In fact, the computational complexity of Edge Colouring for split graphs, or equivalently, $(2P_2, C_4, C_5)$-free graphs [10] and for $P_4$-free graphs has yet to be settled, despite the efforts towards solving the problem for these graph classes [6,8,21].

On a side note, a graph is $k$-edge colourable if and only if its line graph is $k$-vertex colourable. In contrast to the situation for Edge Colouring, the computational complexity of Vertex Colouring has been fully classified for $H$-free graphs [19]. However, the computational complexity for $k$-Vertex Colouring restricted to $H$-free graphs has not been fully classified. It is known that for every $k \geq 3$, $k$-Vertex Colouring on $H$-free graphs is NP-complete if $H$ contains a cycle [9] or an induced claw [14,20], but the case where $H$ is a linear forest has not been settled yet. The complexity status of $k$-Vertex Colouring is even still open for $P_t$-free graphs. More precisely, it is known that the cases $k \leq 2$, $t \geq 1$ (trivial), $k \geq 3$, $t \leq 5$ [13], $k = 3$, $6 \leq t \leq 7$ [2] and $k = 4$, $t = 6$ [7] are polynomial-time solvable and that the cases $k = 4$, $t \geq 7$ [15] and $k \geq 5$, $t \geq 6$ [15] are NP-complete. However, the remaining cases, that is, the cases where $k = 3$ and $t \geq 8$ are still open. We refer to the survey [11] or some recent papers [12,18,25] for further background information.


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