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Graphs vertex-partitionable into strong cliques

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A B S T R A C T

A clique in a graph is strong if it intersects all maximal independent sets. A graph is localizable if it has a partition of the vertex set into strong cliques. Localizable graphs were introduced by Yamashita and Kameda in 1999 and form a rich class of well-covered graphs that coincides with the class of well-covered graphs within the class of perfect graphs. In this paper, we give several equivalent formulations of the property of localizability and develop polynomially testable characterizations of localizable graphs within three non-perfect graph classes: triangle-free graphs, C_4-free graphs, and line graphs. Furthermore, we use localizable graphs to construct an infinite family of counterexamples to a conjecture due to Zaare-Nahandi about k-partite well-covered graphs having all maximal cliques of size k.

1. Introduction

1.1. Background

A *clique* (resp., *independent set*) in a graph is a set of pairwise adjacent (resp., pairwise non-adjacent) vertices. A clique (resp., independent set) in a graph is said to be *maximal* if it is not contained in any larger clique (resp., independent set), and *strong* if it intersects every maximal independent set (resp., every maximal clique). In other words, a strong clique is a transversal of the family of all maximal independent sets of the graph (in fact, an *exact* transversal, since a clique can intersect an independent set in at most one vertex). The notions of strong cliques and strong independent sets in graphs have given rise to several interesting graph classes studied in the literature, including strongly perfect graphs [3,4], very strongly perfect graphs [10,33,45], general partition graphs [38,44], and CIS graphs [6,19] (see also [7]). It is co-NP-complete to test whether a given clique in a graph is strong [65] and NP-hard to test whether a given graph contains a strong clique [34]. On the other hand, the complexity of recognizing: (i) strongly perfect, (ii) general partition, and (iii) CIS graphs is, to the best of our knowledge, open. In terms of strong cliques, these problems are equivalent to testing if: (i) every induced subgraph of a given graph has a strong clique, (ii) every edge of a given graph is contained in a strong clique, and (iii) every maximal clique in a given graph is strong.

In this paper we study two interrelated graph classes, the class of well-covered graphs and the class of localizable graphs. A graph is *well-covered* if all its maximal independent sets have the same size and *localizable* if it has a partition of the vertex set into strong cliques. Well-covered graphs were introduced by Plummer in 1970 [50] and have been studied extensively in the literature, for various reasons. A most obvious reason is that the maximum independent set problem, which is generally NP-complete, can be solved in polynomial time in the class of well-covered graphs by a greedy algorithm.

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Furthermore, well-covered graphs have applications in distributed systems [63], are related to the so-called Generalized Kayles game [22,24], and play an important role in commutative algebra, where they are typically referred to as *unmixed* graphs, see, e.g., [32,47,61]. The well-coveredness property of a graph G is equivalent to the property that the simplicial complex of the independent sets of G is *pure* and generalizes the algebraically defined concept of a Cohen–Macaulay graph (see, e.g., [13]). For further background on well-covered graphs, we refer to the surveys by Plummer [51] and Hartnell [30].

It is not difficult to see that every localizable graph is well-covered. However, unlike well-covered graphs, localizable graphs have not been much studied in the literature. They were introduced in 1999 by Yamashita and Kameda [63] and appeared, at least implicitly, in other works, for example in 1983 in the work of Finbow and Hartnell [24, Proposition 1] and more recently, in 2015, the work of Zaare-Nahandi [64, Theorem 2.1]. In particular, Finbow and Hartnell proved in [24] that every well-covered graph of girth at least 8 has a perfect matching formed by pendant edges. Since every pendant edge is a strong clique, this implies that every well-covered graph of girth at least 8 is localizable. Unaware of this result, Yamashita and Kameda observed in [63] that all well-covered trees are localizable, pointed out that the converse inclusion fails in general, and asked for a characterization of localizable graphs.

1.2. Aims and motivation

In this work we initiate a systematic study of localizable graphs. Our motivations for this study are threefold.

First, the class of localizable graphs is defined via the notion of strong cliques in graphs, which, as outlined above, gives rise to many interesting and challenging graph classes. It is also worth pointing out that strong cliques can arise as images, under suitable transformations, of other well-studied objects in graphs; for example, transforming a regular triangle-free graph to the complement of its line graph maps any perfect matching into a strong clique (see [5,46] for applications of this observation). All these facts lead us to believe that the class of localizable graphs is worthy of study on its own and that investigating it is likely to lead to interesting new results and questions on connections between various graph notions, inclusions between graph classes, and complexity considerations. Indeed, some of the characterizations obtained in this paper unify and generalize several known results from the literature. Furthermore, localizable graphs will be a useful tool in our construction of counterexamples to a conjecture due to Zaare-Nahandi [64] about k-partite well-covered graphs having all maximal cliques of size k.

Second, a better understanding of the class of localizable graphs may lead to further insights on the class of well-covered graphs. Since every localizable graph is well-covered, any construction leading to localizable graphs will immediately result in a family of well-covered graphs. Moreover, as we will see in Section 2, the two classes coincide within the class of perfect graphs, and, more generally, within the class of semi-perfect graphs, defined as graphs in which the clique cover number and the independence number coincide [64]. This relation may lead to new perspectives and insights into classes of well-covered perfect and semi-perfect graphs.

Finally, the study of localizable graphs directly addresses the question about characterizations of localizable graphs raised by Yamashita and Kameda. As we will show in Section 2, it is generally NP-hard to determine if a given graph is localizable. This motivates the question of identifying graph classes where the property of localizability can be tested efficiently. Several examples of such graph classes will be presented in this paper.

1.3. Overview of results

Our results can be roughly divided into three main parts, which we now summarize.

1. Equivalent formulations. We give several equivalent formulations of localizability, which allow us to derive several characterizations of localizable graphs within the class of semi-perfect graphs.

2. Counterexamples to a conjecture due to Zaare-Nahandi. We use localizable graphs to construct an infinite family of counterexamples to a conjecture due to Zaare-Nahandi about k-partite well-covered graphs having all maximal cliques of size k.

Conjecture 1 (*Zaare-Nahandi [64]*)**.** *Let G be a k-partite well-covered graph in which all maximal cliques are of size k. Then G is semi-perfect.*

A graph is *co-well-covered* if its complement is well-covered. We will show that Conjecture 1 can be equivalently posed in terms of localizable graphs, as follows.

Conjecture 2. *Let G be a localizable co-well-covered graph. Then $\overline{G}$ is localizable.*

We disprove these two equivalent conjectures by constructing an infinite family of counterexamples to the weaker statement saying that every localizable co-well-covered graph has a strong independent set. We also give a related hardness result showing that it is NP-hard to determine whether the complement of a given localizable co-well-covered graph is localizable. The proof is based on a reduction from the 3-colorability problem in triangle-free graphs and shows a way how to transform, in a simple way, any triangle-free graph of chromatic number at least four to a counterexample to Conjecture 1.

3. Characterizations. We characterize localizable graphs within the classes of triangle-free graphs and C_4-free graphs. Both characterizations lead to polynomial-time recognition algorithms. To put these results in perspective, note that no characterization of well-covered triangle-free or C_4-free graphs is known. (Well-covered graphs of girth at least five were characterized by Finbow et al. [23] and well-covered graphs without a subgraph isomorphic to C_4 were studied by Brown et al. [9].) The characterizations of localizable triangle-free and C_4-free graphs generalize, respectively, a characterization due to Dean and Zito about well-covered triangle-free semi-perfect graphs [18] and characterizations due to Prisner et al. [52] and Dean and Zito [18] about well-covered chordal, resp. C_4-free semi-perfect graphs.

We develop a characterization of localizable graphs within the class of line graphs. This characterization is more involved than the characterizations of localizable triangle-free and C_4-free graphs and also leads to a polynomial-time recognition algorithm. Note that the line graph of a graph G is well-covered if and only if G is *equimatchable*, that is, if all maximal matchings of G are of the same size (see, e.g., [41,42]). Our characterization of localizable line graphs is derived independently of the characterizations of equimatchable graphs and implies, in particular, that the equimatchable bipartite graphs are the only triangle-free graphs whose line graphs are localizable.

2. Equivalent formulations of localizability

In this section, we give several equivalent formulations of the property of localizability. First we recall some definitions and fix some notation. We consider only finite, simple and undirected graphs. Given a graph $G = (V, E)$, its complement $\overline{G}$ is the graph with vertex set V in which two distinct vertices are adjacent if and only if they are non-adjacent in G. By K_n, P_n, and C_n we denote the n-vertex complete graph, path, and cycle, respectively, and by $K_{m,n}$ the complete bipartite graph with parts of sizes m and n. The *degree* of a vertex v in a graph G is denoted by $d_G(v)$, its neighborhood by $N_G(v)$ (or simply by $N(v)$ if the graph is clear from the context), and its closed neighborhood by $N_G[v]$ (or simply by $N[v]$). For a set of vertices $X \subseteq V(G)$, we denote by $N_G(X)$ (or $N(X)$) the set of all vertices in $V(G) \setminus X$ having a neighbor in X. A *triangle* in a graph is a clique of size 3; a graph is *triangle-free* if it has no triangles. Similarly, a graph is *C_4-free* if it has no induced subgraph isomorphic to a C_4. We will often identify a triangle with the set of its edges; whether we consider a triangle as a set of vertices or as a set of edges will always be clear from the context. For graph theoretic notions not defined here, we refer the reader to [62].

Given a graph G, we denote by $\alpha(G)$ its *independence number*, that is, the maximum size of an independent set in G, by $i(G)$ its *independent domination number*, that is, the minimum size of an independent dominating set in G (equivalently: the minimum size of a maximal independent set in G), by $\omega(G)$ its *clique number*, that is, the maximum size of a clique in G, by $\chi(G)$ its *chromatic number*, that is, the minimum number of independent sets that partition its vertex set, and by $\theta(G)$ its *clique cover number*, that is, the minimum number of cliques that partition its vertex set. Every graph G has $\alpha(G) = \omega(\overline{G})$, $\theta(G) = \chi(\overline{G})$, $\chi(G) \geq \omega(G)$, and $\theta(G) \geq \alpha(G)$. It follows that every graph G satisfies the following chain of inequalities:

$$i(G) \leq \alpha(G) \leq \theta(G)\,. \tag{1}$$

Clearly, a graph G is well-covered if and only if $i(G) = \alpha(G)$.

For a positive integer k, we say that a graph G is *k-localizable* if it admits a partition of its vertex set into exactly k strong cliques. Recall that a graph G is said to be *semi-perfect* if $\theta(G) = \alpha(G)$, that is, if there exists a collection of $\alpha(G)$ cliques partitioning its vertex set. We will refer to such a collection as an *α-clique cover* of G. Thus, G is semi-perfect if and only if it has an α-clique cover, and G is $\alpha(G)$-localizable if and only if it has an α-clique cover in which every clique is strong.

Now we have everything ready to prove the following equivalent formulations of localizability.

Theorem 2.1. *For every graph G, the following statements are equivalent.*

(a) *G is localizable.*
(b) *G is $\alpha(G)$-localizable (equivalently, G has an α-clique cover in which every clique is strong).*
(c) *G has an α-clique cover and every clique in every α-clique cover of G is strong.*
(d) *G is well-covered and semi-perfect.*
(e) *$i(G) = \theta(G)$.*

Proof. (a) ⇒ (b): Suppose that G is localizable, and let $C_1, \ldots, C_k$ be a collection of strong cliques of G partitioning its vertex set. Let S be a maximal independent set of G. Then S intersects each C_i in a vertex, which implies that $|S| = \sum_{i=1}^{k} |C_i \cap S| = k$. Since S was arbitrary, G is well-covered, with $\alpha(G) = k$. In particular, G is $\alpha(G)$-localizable. Thus, (a) implies (b).

(b) ⇒ (d): Suppose that G is $\alpha(G)$-localizable. Then $\theta(G) \leq \alpha(G)$. Since the opposite inequality holds for every graph, we conclude that $\theta(G) = \alpha(G)$, that is, G is semi-perfect. Moreover, since $V(G)$ has a partition into $\alpha(G)$ strong cliques, every maximal independent set in G is of size $\alpha(G)$. Thus, G is also well-covered and the implication (b) ⇒ (d) follows.

(d) ⇒ (c): Suppose that G is well-covered and semi-perfect. Since G is semi-perfect, it has an α-clique cover. Now consider an arbitrary α-clique cover $C_1, \ldots, C_{\alpha(G)}$ of G. We will show that each clique C_i is strong. Suppose this is not the case. Without loss of generality, assume that C_1 is not strong. Then, there exists a maximal independent set S of G disjoint from C_1. Consequently, $|S| = \sum_{i=2}^{\alpha(G)} |S \cap C_i| \leq \alpha(G) - 1$, contrary to the fact that G is well-covered. This proves that every C_i is strong and establishes the implication (d) ⇒ (c).

(c) ⇒ (a): Trivial.

(d) ⇔ (e): Recall that G is well-covered if and only if $i(G) = \alpha(G)$. Therefore, condition (d) is equivalent to $i(G) = \alpha(G) = \theta(G)$, which, by (1), is equivalent to condition (e).

Altogether, the above equivalence and implications complete the proof of the proposition. □

Given two sets of vertices A, B in a graph G, we say that A *dominates* B if, for every vertex v in B, either v is in A or v has a neighbor in A. Theorem 2.1 generalizes the following result of Zaare-Nahandi about semi-perfect graphs.

Theorem 2.2 (*Theorem 2.1 in [64]*). *Let $G = (V, E)$ be a semi-perfect graph with an α-clique cover $Q_1, \ldots, Q_{\alpha(G)}$. Then G is well-covered if and only if, for each $i \in \{1, \ldots, \alpha(G)\}$, the following holds: if $A \subseteq V \setminus Q_i$ dominates Q_i, then A is not an independent set.*

To see why Theorem 2.1 implies Theorem 2.2, note first that since every semi-perfect graph has an α-clique cover, Theorem 2.1 specialized to the case of semi-perfect graphs can be stated as follows.

Theorem 2.3. *For every semi-perfect graph G, the following statements are equivalent.*

(a) *G is localizable.*
(b) *G is $\alpha(G)$-localizable (equivalently, G has an α-clique cover in which every clique is strong).*
(c) *Every clique in every α-clique cover of G is strong.*
(d) *G is well-covered.*
(e) *$i(G) = \theta(G)$.*

Next, observe that a clique C is strong if and only if it is not dominated by an independent set $I \subseteq V \setminus C$. Indeed, if C is not strong, then any maximal independent set disjoint from C dominates C. If some independent set $I \subseteq V \setminus C$ dominates C, then any maximal independent set I' containing I is disjoint from C. Indeed, since I dominates C, no vertex of C can be added to I without violating the condition that the set is independent. It follows that C is not strong whenever some independent set $I \subseteq V \setminus C$ dominates C. Theorem 2.2 can be therefore equivalently stated as follows:

Theorem 2.4. *Let $G = (V, E)$ be a semi-perfect graph with an α-clique cover $Q_1, \ldots, Q_{\alpha(G)}$. Then G is well-covered if and only if all cliques Q_i are strong.*

We conclude that Theorem 2.4 follows from the equivalencies between statements (b), (c), and (d) in Theorem 2.3.
We will use Theorem 2.1 several times in the paper. The following consequence will also be of use.

Corollary 2.5. *A semi-perfect graph is well-covered if and only if it is localizable.*

A graph G is *perfect* if $\chi(H) = \omega(H)$ holds for every induced subgraph H of G. Since the complementary graph $\overline{G}$ is perfect whenever G is perfect [43], every perfect graph G is also semi-perfect. In particular, a perfect graph is well-covered if and only if it is localizable. A graph is *weakly chordal* if neither G nor its complement contains an induced cycle of length at least 5. Sankaranarayana and Stewart [56] and Chvátal and Slater [16] proved that the problem of recognizing well-covered graphs is co-NP-complete, even for weakly chordal graphs. Hayward [31] proved that weakly chordal graphs are perfect, and consequently, they are also semi-perfect. Based on this fact, Corollary 2.5 implies that a weakly chordal graph is well-covered if and only if it is localizable. In particular, the problem of determining whether a given weakly chordal graph is localizable is co-NP-complete.

3. Counterexamples to a conjecture by Zaare-Nahandi

In [64], Zaare-Nahandi posed the following conjecture. Recall that a graph G is said to be semi-perfect if $\theta(G) = \alpha(G)$.

Conjecture 1 (*Restated*). *Let G be a k-partite well-covered graph in which all maximal cliques are of size k. Then G is semi-perfect.*

The conjecture can be equivalently stated in terms of localizable graphs as follows.

Conjecture 2 (*Restated*). *Let G be a localizable co-well-covered graph. Then $\overline{G}$ is localizable.*

Proposition 3.1. *Conjectures 1* and 2 *are equivalent.*

Proof. First we prove that Conjecture 1 implies Conjecture 2. Assume the validity of Conjecture 1. Let G be a localizable co-well-covered graph. By Theorem 2.1, this is equivalent to: G and $\overline{G}$ are well-covered and $\theta(G) = \alpha(G)$. Let $k = \alpha(G)$. Then $\chi(\overline{G}) = k$, hence $\overline{G}$ is a k-partite well-covered graph in which all maximal cliques are of size k. By Conjecture 1, $\overline{G}$ is semi-perfect, that is, $\theta(\overline{G}) = \alpha(\overline{G})$. Therefore, since $\overline{G}$ is well-covered, $\overline{G}$ is localizable by Theorem 2.1. Thus, Conjecture 2 holds.

Conversely, Conjecture 2 implies Conjecture 1. Assume the validity of Conjecture 2. Let G be a k-partite well-covered graph in which all maximal cliques are of size k. Then $\chi(G) = \omega(G) = k$ and all the maximal independent sets of $\overline{G}$ are of size k. These conditions imply that $\theta(\overline{G}) = \alpha(\overline{G})$ and both G and $\overline{G}$ are well-covered. By Theorem 2.1, this is equivalent to: $\overline{G}$ is a localizable co-well-covered graph. Conjecture 2 implies that G is localizable, and by Theorem 2.1, G is semi-perfect. Thus, Conjecture 1 holds. □

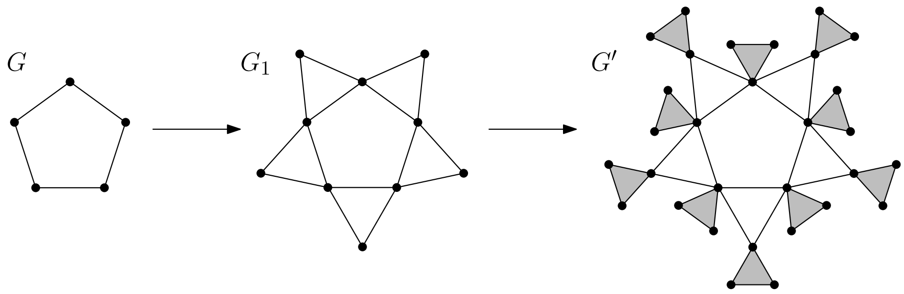

Fig. 1. An example of the transformation $G \to G'$. The simplicial cliques of G' (which partition $V(G')$) are shaded gray.

In what follows, we disprove Conjecture 1 in two different ways. First, we give an infinite family of counterexamples showing that even the following conjecture, which would be a consequence of Conjectures 1 and 2, fails.

Conjecture 3. *Let G be a localizable co-well-covered graph. Then $\overline{G}$ has a strong clique.*

In the proof of the following theorem, we will use the notion of simplicial cliques. A clique C in a graph G is said to be *simplicial* if there exists a vertex $v \in V(G)$ such that $C = N[v]$, the closed neighborhood of v. Observe that every simplicial clique is strong.

Theorem 3.2. *Conjecture 3 is false. Consequently, Conjectures 1 and 2 are false.*

Proof. Let C be an odd cycle of length at least 5, and let G be the corona of C, that is, the graph obtained from C by adding to it $|V(C)|$ new vertices, each adjacent to a different vertex of cycle (and there are no other edges in G). Each of the $|V(C)|$ sets $N[v]$, where $v \in V(G) \setminus V(C)$, is a simplicial (hence strong) clique in G. These cliques are pairwise disjoint and their union is $V(G)$. Therefore, G is localizable. Since every maximal clique of G has size 2, we infer that G is co-well-covered. We claim that $\overline{G}$ does not have any strong clique, which is equivalent to stating that G does not have any strong independent set. This is true since any strong independent set in G would have to contain a vertex from each of the edges of C (as these edges are all maximal cliques in G); however, as C is not bipartite, no independent set of C hits all edges of C. □

Second, we give a 'computational complexity' disproof of Conjecture 1, by showing that not only there are localizable co-well-covered graphs whose complement is not localizable, but that it is NP-hard to determine if the complement of a given localizable co-well-covered graph is localizable.

Theorem 3.3. *Given a localizable co-well-covered graph G, it is NP-hard to determine if $\overline{G}$ is localizable.*

Proof. We make a reduction from the 3-Colorability problem in triangle-free graphs, which is NP-hard [37]. Given a triangle-free graph G, we will construct from G a localizable co-well-covered graph G' such that G is 3-colorable if and only if $\overline{G'}$ is localizable.

The graph G' is obtained from G in two steps. First, every edge of G is extended into a triangle with a unique new vertex; let G_1 be the resulting graph. Second, to every vertex v of G_1 two new vertices are added, say v' and v'', which are adjacent to each other and to v. Then, G' is the so obtained graph. An example of the reduction is shown in Fig. 1.

Since there is a set of $|V(G_1)|$ simplicial cliques partitioning $V(G')$, the graph G' is localizable. Since G is triangle-free, every maximal clique of G' is of size 3, therefore G' is co-well-covered. To complete the proof, we show that G is 3-colorable if and only if the complement of G' is localizable. Suppose that G is 3-colorable, and let c be a 3-coloring of G. Extend c to a 3-coloring c' of G'. The color classes of c' define a partition of $V(G')$ into three independent sets, equivalently, a partition of the vertex set of the complement of G' into three cliques, say C_1, C_2, C_3. Since all maximal cliques of G' are of size 3, every color class contains a vertex of each maximal clique of G, which means that each C_i is a strong clique in the complement of G'. Therefore the complement of G' is localizable. Conversely, since $\alpha(\overline{G'}) = \omega(G') = 3$, if the complement of G' is localizable, then it has a partition of its vertex into 3 strong cliques, and therefore G' is 3-colorable. But then so is G, as an induced subgraph of G'. □

It follows from the proof of Theorem 3.3 that if G is any triangle-free graph of chromatic number at least 4, then the graph G' obtained from G as in the above proof (cf. Fig. 1) is a localizable co-well-covered graph with a non-localizable complement. There are many known constructions of triangle-free graphs of high chromatic number, see, e.g., [1,48,66].

We would also like to point out that in [64] Zaare-Nahandi claimed that the problem of determining if a given semi-perfect graph is well-covered is a polynomially solvable task. However, a polynomial-time recognition algorithm for testing if a given semi-perfect graph is well-covered would imply P = NP. This follows from the fact that the recognition problem of well-covered weakly chordal graphs is co-NP-complete and that every weakly chordal graph is semi-perfect.[1]

4. Characterizations

The fact the problem of recognizing localizable graphs is NP-hard motivates the question of identifying graph classes where the property of localizability can be tested efficiently. In this section, we characterize localizable graphs within the classes of triangle-free graphs, C_4-free graphs, and line graphs. Our characterizations imply polynomial-time recognition algorithms of localizable graphs within each of these classes.

4.1. Overview

First, we summarize known results from the literature leading immediately to characterizations of localizable graphs in various graph classes, some of which lead to polynomial-time recognition algorithms. For the definitions of graph classes mentioned below and further background, we refer to [8,25].

By Corollary 2.5, a semi-perfect graph is localizable if and only if it is well covered. Therefore, any result characterizing well-covered graphs within a class of semi-perfect graphs immediately implies the same characterization of localizable graphs within the class. This yields characterizations of localizable graphs within the classes of bipartite graphs [54], or, more generally, of triangle-free semi-perfect graphs [18]. Moreover, the class of localizable graphs can be recognized in polynomial time in any class of semi-perfect graphs for which the well-coveredness property can be tested in polynomial time. Examples of classes of perfect graphs for which the well-coveredness (and thus localizability) property can be tested in polynomial time include perfect graphs of bounded degree [12] and claw-free perfect graphs [59,60]. Chordal well-covered (equivalently: chordal localizable) graphs were characterized by Prisner et al. [52], as follows. Prisner et al. showed that a chordal graph is well-covered if and only if each vertex is in a unique simplicial clique. (Recall that a clique is simplicial if it consists of a vertex and all its neighbors.) They also proved that the same condition characterizes well-covered graphs among *simplicial* ones, that is, among graphs in which every vertex is in a simplicial clique. Since each simplicial clique is strong, this property implies localizability, and therefore localizable simplicial graphs can also be recognized in polynomial time. The characterization of well-covered chordal graphs due to Prisner et al. was further generalized by Dean and Zito, who showed in [18, Theorem 4.2] that a C_4-free semi-perfect graph G is well-covered if and only if every minimum clique cover $\mathcal{C}$ of G is a partition of the vertex set and every clique of $\mathcal{C}$ contains a simplicial vertex. It is not difficult to see that this condition is equivalent to the condition that each vertex is in a unique simplicial clique.

By Theorem 2.1, a graph is localizable if and only if its independent domination and clique cover numbers coincide. Therefore, the class of localizable graphs can be recognized in polynomial time in any class of graphs for which these two parameters are polynomially computable. Examples of such graph classes include the class of circular-arc graphs [14,36], graphs of bounded clique-width [17,49,53], and, using the fact that the clique cover number is polynomially computable in the class of perfect graphs [27], any class of perfect graphs for which the independent domination problem is polynomially solvable, for instance distance-hereditary graphs [15] (which are of clique-width at most 3 [26]) and cocomparability graphs [39]. Note also that well-coveredness (equivalently: localizability) of cocomparability graphs is equivalent to a known (and polynomially verifiable) property of a derived partially ordered set. For a graph G, let us denote by $\mathcal{P}_G$ the set of all partial orders (posets) with ground set $V(G)$ in which two distinct elements are incomparable if and only if they are adjacent in G.

[1] Let us explain what we believe was the error in the arguments from [64] leading the author to conclude that the problem of determining if a given semi-perfect graph is well-covered is a polynomially solvable task. The author assumes that a semi-perfect graph G is given together with an α-clique cover $Q_1, \ldots, Q_k$ where $k = \alpha(G)$, denotes by $N(Q_i)$ the set of vertices that are either in Q_i or have a neighbor in Q_i and writes:

"*... Therefore, checking well-coveredness of the graph G is equivalent to checking that, for each i, $1 \le i \le k$, the set of vertices of Q_i is part of a minimal vertex cover of G. But, this is a simple task: it is enough to check that the set of vertices of Q_i is a minimal vertex cover of the subgraph of G induced by $N(Q_i)$, which can be done in polynomial time.*"

Recall that a *vertex cover* in a graph G is a set S of vertices such that every edge of G has at least one endpoint in S and that a set $S \subseteq V(G)$ is a (minimal) vertex cover if and only if $V(G) \setminus S$ is a (maximal) independent set. Thus, a clique Q_i is contained in a minimal vertex cover S if and only if Q_i is disjoint from some maximal independent set, which is equivalent to the statement that Q_i is not strong. Therefore, the first sentence in the above text is correct, in the sense that G is not well-covered if and only if some Q_i is part of a minimal vertex cover of G (cf. Theorem 2.4). It is also true that the condition that Q_i is a minimal vertex cover of the subgraph of G induced by $N(Q_i)$ can be tested in polynomial time. However, this condition is not equivalent to the condition that Q_i is part of a minimal vertex cover of G. This latter condition is equivalent to testing if Q_i is disjoint from some maximal independent set, which, in general, is an NP-complete problem [65], and remains NP-complete for weakly chordal and thus for semi-perfect graphs. This can be seen by analyzing the NP-hardness proofs of recognizing well-covered graphs due to Sankaranarayana and Stewart [56] and Chvátal and Slater [16]. It is also not difficult to construct a concrete counterexample to the claimed equivalence between the condition that Q_i is a minimal vertex cover of the subgraph of G induced by $N(Q_i)$ and the condition that Q_i is part of a minimal vertex cover of G. For example, let G be the graph with vertex set $\{a, b, c, d, e, f\}$ and edge set $\{ab, ac, bc, bd, cd, de, ef\}$, and consider an α-clique cover Q_1, Q_2, Q_3 of G where $Q_1 = \{a, b, c\}$, $Q_2 = \{d, e\}$, and $Q_3 = \{f\}$. Then, Q_2 is part of a minimal vertex cover of G, for example, of $\{a, b, d, e\}$, but Q_2 is not a minimal vertex cover (in fact, not even a vertex cover) of the subgraph of G induced by $N(Q_2) = \{b, c, d, e, f\}$.

A graph G is cocomparability if and only if $\mathcal{P}_G \neq \emptyset$. A cocomparability graph G is well-covered if and only if some poset in $\mathcal{P}_G$ is *graded* (that is, all its maximal chains are of the same size), if and only if all posets in $\mathcal{P}_G$ are graded.

Finally, let us note that the classification of well-covered cubic graphs due to Campbell et al. [11] implies a classification of localizable cubic graphs. The class of localizable cubic graphs consists of an infinite family of planar cubic graphs along with three small graphs (K_4, $K_{3,3}$, and the complement of C_6).

We summarize the above observations in the following theorem.

Theorem 4.1. *The problem of recognizing localizable graphs is polynomially solvable within each of the following graph classes: cubic graphs, C_4-free semi-perfect graphs, cocomparability graphs, perfect graphs of bounded degree, claw-free perfect graphs, simplicial graphs, and graphs of bounded clique-width.*

The results obtained in the rest of this section allow to update the above list by replacing the class of C_4-free semi-perfect graphs with the larger class of C_4-free graphs and adding to the list the classes of triangle-free graphs and line graphs.

4.2. Triangle-free graphs and C_4-free graphs

Let G be a triangle-free graph without isolated vertices. Since G is triangle-free and without isolated vertices, G is localizable if and only if G has a partition of its vertex set into strong cliques of size two.

Proposition 4.2. *For a triangle-free graph G without isolated vertices, the following properties are equivalent:*

1. *G is localizable.*
2. *G has a perfect matching every edge of which is a strong clique.*
3. *G has a perfect matching, and every edge of every perfect matching in G is a strong clique.*

Proof. Since G is triangle-free and without isolated vertices, every strong clique is of size two.

Implications $3 \Rightarrow 2 \Rightarrow 1$ hold for all graphs. The first one is trivial and the second one follows directly from the definition of localizability. Suppose now that G is localizable. By Theorem 2.1, G has an α-clique cover and every clique in every α-clique cover of G is strong. Let $\mathcal{C} = \{C_1, \ldots, C_{\alpha(G)}\}$ be an α-clique cover of G. Since every clique C_i is strong, every C_i is of size two, that is, $\mathcal{C}$ is a perfect matching in G; moreover, $\alpha(G) = |\mathcal{C}| = |V(G)|/2$. Conversely, if M is a perfect matching in G, then M is an α-clique cover of G, therefore, since every clique in every α-clique cover is strong, every edge of M is a strong clique. □

Proposition 4.2 and Corollary 2.5 imply the characterization of well-covered graphs within the class of triangle-free semi-perfect graphs by Dean and Zito [18, Theorem 4.3]. From the algorithmic point of view, Proposition 4.2 implies that there is a polynomial-time algorithm to test if a given triangle-free graph is localizable: After the removal of isolated vertices, one can use Edmonds' algorithm [20] to test if the graph has a perfect matching, and if a perfect matching is found, each of its edges is tested for being a strong clique. It is not difficult to see that an edge $\{u, v\}$ in a triangle-free graph G is a strong clique if and only if every vertex of $N(u)$ is adjacent to every vertex of $N(v)$.

Remark 4.3. The two equivalent properties characterizing localizability in the class of triangle-free graphs without isolated vertices appeared in the literature in various contexts. For example, as shown independently by Staples in 1975 [58, Theorem 1.11] and by Favaron in 1982 [21], the two properties exactly characterize the class of very well-covered graphs. (A graph G is said to be *very well-covered* if it has no isolated vertices and every maximal independent set is of size $|V(G)|/2$). Clearly, if a graph G has a perfect matching every edge of which is a strong clique, then $V(G)$ can be partitioned into strong cliques, and thus G is localizable. It follows that every very well-covered graph is localizable. Condition 2 was also studied in the context of commutative algebra by Benedetti and Varbaro [2].

We now turn to a characterization of C_4-free localizable graphs. First, we characterize strong cliques in a C_4-free graph.

Lemma 4.4. *A clique in a C_4-free graph is strong if and only if it is simplicial.*

Proof. Recall that every simplicial clique is strong. For the converse direction, let C be a strong clique in a C_4-free graph G. Suppose for a contradiction that C is not simplicial. Then, every vertex of C has a neighbor in $V(G) \setminus C$, that is, $N(C)$ dominates C. Let I be any minimal set of vertices in $N(C)$ that dominates C. We claim that I is an independent set in G. Suppose that there exists a pair x, y of adjacent vertices in I. The C_4-freeness of G implies that the neighborhoods of x and y in C are comparable, that is, $N(x) \cap C \subseteq N(y) \cap C$ or $N(y) \cap C \subseteq N(x) \cap C$. But then, assuming (w.l.o.g.) $N(x) \cap C \subseteq N(y) \cap C$, we could remove x from I to obtain a subset of $N(C)$ that dominates C properly contained in I, contradicting the minimality of I. This shows that I is independent, as claimed. Extending I to an arbitrary maximal independent set of G yields a maximal independent set disjoint from C, contradicting the fact that C is a strong clique. The obtained contradiction completes the proof that C is simplicial. □

Theorem 4.5. *A C_4-free graph G is localizable if and only if each vertex of G is in a unique simplicial clique.*

Proof. By Lemma 4.4, G is localizable if and only if its vertex set can be partitioned into simplicial cliques. This condition is clearly satisfied if each vertex is in a unique simplicial clique.

Suppose now that $V(G)$ partitions into simplicial cliques $C_1, \ldots, C_k$. It suffices to show that for every $v \in C_i$, clique C_i is the only simplicial clique containing v. This is clear if v is a simplicial vertex (in this case C_i is the only maximal clique containing v). Suppose now that v is not a simplicial vertex and that C' is a simplicial clique containing v such that $C' \neq C_i$. Let $v' \in C'$ be a vertex such that $N[v'] = C'$. Consider the simplicial clique C_j such that $v' \in C_j$. We have $j \neq i$, since otherwise we would have $C_i \subseteq N[v'] = C'$, contrary to the fact that C_i and C' are distinct maximal cliques. Since C_j is a maximal clique, vertex v has a non-neighbor in C_j, say v''. But now, vertices v and v'' form a pair of non-adjacent neighbors of v', contrary to the fact that the neighborhood of v' is a clique. This shows that C_i is the only simplicial clique containing v and completes the proof. □

Since the set of simplicial cliques in a graph can be computed in polynomial time, Theorem 4.5 implies the existence of a polynomial-time algorithm to determine if a given C_4-free graph is localizable. Moreover, since within the class of semi-perfect graphs, localizable graphs coincide with well-covered ones, Theorem 4.5 generalizes the characterizations of well-covered graphs within the classes of chordal and of C_4-free semi-perfect graphs due to Prisner et al. [52] and Dean and Zito [18], respectively.

To put the results of Proposition 4.2 and Theorem 4.5 in perspective, recall that no characterizations of well-covered triangle-free or C_4-free graphs are known.

4.3. Line graphs

Consider a graph $H = (V_H, E_H)$ and let $L(H)$ be the graph constructed as follows: we associate a vertex with each edge in E_H and two vertices are adjacent if and only if the corresponding edges in H share an endpoint. A graph G is said to be a *line graph* if $G = L(H)$ for some graph H. In this section we characterize localizable line graphs. The characterization implies the existence of a polynomial-time algorithm to determine whether a given line graph is localizable.

Recall that the line graph of a graph H is well-covered if and only if H is equimatchable, that is, if all maximal matchings of G are of the same size. The question of characterizing equimatchable graphs was posed by Grünbaum in 1974 [28]; in the same year, equimatchable graphs were studied and characterized by Lewin [42] and shown to be polynomially recognizable by Lesk et al. [41]. Since every localizable graph is well-covered, every graph the line graph of which is localizable is equimatchable. Moreover, since the line graphs of bipartite graphs are perfect (see, e.g., [57]) and hence semi-perfect, Corollary 2.5 implies that the line graph of a bipartite graph H is localizable if and only if it is well-covered; equivalently, if H is equimatchable. Lesk et al. [41] characterized equimatchable bipartite graphs as follows.

Theorem 4.6 (*Lesk et al. [41]*)**.** *A connected bipartite graph H with a bipartition of its vertex set into two independent sets $V(H) = U \cup W$ with $|U| \leq |W|$ is equimatchable if and only if for all $u \in U$, there exists a non-empty set $X \subseteq N(u)$ such that $|N(X)| \leq |X|$.*

Given a graph H, we say that a vertex $v \in V(H)$ is *strong* if every maximal matching of H covers v. The above characterization has the following consequence.

Lemma 4.7. *A connected bipartite graph H is equimatchable if and only if it has a bipartition of its vertex set into two independent sets $V(H) = U \cup W$ such that all vertices of U are strong.*

Proof. First, we show that a vertex $u \in V(H)$ in a bipartite graph H is strong if and only if there exists a non-empty set $X \subseteq N(u)$ such that $|N(X)| \leq |X|$. This is equivalent to showing that u is not strong if and only if $|N(X)| > |X|$ for all non-empty sets $X \subseteq N(u)$, which is further equivalent to the condition that $|N_{H-u}(X)| \geq |X|$ for all non-empty sets $X \subseteq N_H(u)$. By Hall's Theorem [29], this is equivalent to the existence of a matching M in the graph $H - u$ such that every vertex of $N_H(u)$ is incident with an edge in M. This is in turn equivalent to the condition that H contains a maximal matching not covering u, that is, that u is not strong in H, as claimed.

Let H be a connected bipartite graph. Suppose first that H is equimatchable and fix a bipartition of its vertex set into two independent sets $V(H) = U \cup W$ with $|U| \leq |W|$. Theorem 4.6 implies that for all $u \in U$, there exists a non-empty set $X \subseteq N(u)$ such that $|N(X)| \leq |X|$. By the above equivalence, all vertices in U are strong. Conversely, suppose that H is a connected bipartite graph with a bipartition of its vertex set into two independent sets $V(H) = U \cup W$ such that all vertices of U are strong. By the above equivalence, for all $u \in U$, there exists a non-empty set $X \subseteq N(u)$ such that $|N(X)| \leq |X|$. Let M be a maximal matching of H. By assumption on U, every vertex of U is incident with an edge of M. This implies that $|U| = |M| \leq |W|$. Thus, by Theorem 4.6, H is equimatchable. □

As pointed out above, the line graph of every equimatchable bipartite graph is localizable. In what follows, we will show that graphs the line graphs of which is localizable do not differ too much from equimatchable bipartite graphs. To describe the result, we need a couple of definitions.

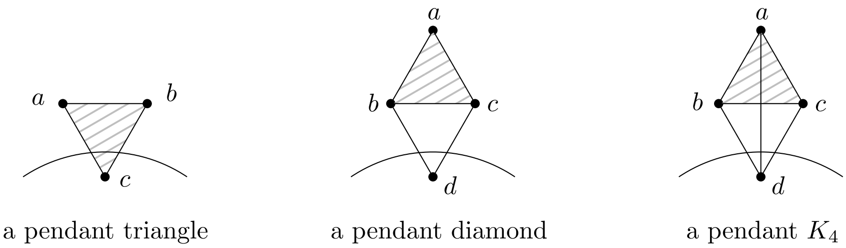

Fig. 2. The three types of pendant subgraphs. The shaded triangles represent a pendant triangle, a triangle contained in a pendant diamond, and a triangle contained in a pendant K_4, respectively.

Definition 4.8. The *diamond* is the graph with vertices a, b, c, d and edges ab, ac, bc, bd, cd; the vertices a and d are its *tips*. Let T be a triangle in a graph H and let a, b, c be the vertices of T. We say that T is:

- a *pendant triangle* of H if $d_H(a) = d_H(b) = 2 < d_H(c)$,
- *contained in a pendant diamond* if H has a subgraph with vertices a, b, c, d inducing a diamond with tips a and d such that $d_H(a) = 2$ and $d_H(b) = d_H(c) = 3 \leq d_H(d)$, and
- *contained in a pendant* K_4 if H has a subgraph with vertices a, b, c, d inducing a K_4 such that $d_H(a) = d_H(b) = d_H(c) = 3 < d_H(d)$.

Any diamonds and K_4s as above will be referred to as *pendant diamonds* and *pendant* K_4*s* (of H), respectively. Pendant triangles, pendant diamonds, and pendant K_4s of H will be referred to briefly as its *pendant subgraphs*; see Fig. 2. Note that if a graph H has a pendant subgraph, then H is not isomorphic to any graph in the set $\{K_3, K_4, \text{diamond}\}$. Moreover, every pendant subgraph of H has a unique *root*, that is, a vertex connecting it to the rest of the graph. We denote by $R_{\triangle}(H)$, $R_{\diamond}(H)$, and $R_{\oplus}(H)$ the sets of roots of all pendant triangles, pendant diamonds, and pendant K_4s of H, respectively. The *pendant reduction* of H is the graph obtained from H by deleting from H all non-root vertices of its pendant subgraphs.

The characterization of localizable line graphs is given by the following theorem.

Theorem 4.9. *Let H be a connected graph and let $G = L(H)$. Then, G is localizable if and only if one of the following holds:*

1. *H is not isomorphic to any graph in the set $\{K_3, K_4, \text{diamond}\}$ and its pendant reduction F is a connected bipartite graph with a bipartition of its vertex set into two independent sets U and W such that*
   - (a) *$R_{\diamond}(H) \cup R_{\oplus}(H) \subseteq U$,*
   - (b) *$R_{\triangle}(H) \subseteq W$, and*
   - (c) *each vertex $u \in U \setminus R_{\oplus}(H)$ is strong in the graph $F - (N_F(u) \cap R_{\triangle}(H))$.*
2. *H is isomorphic to either K_3 or K_4.*

Before proving Theorem 4.9, let us note two of its consequences.

Corollary 4.10. *Let H be a connected triangle-free graph and let $G = L(H)$. Then, G is localizable if and only if H is an equimatchable bipartite graph.*

Proof. If H is triangle-free, then its pendant reduction equals H and $R_{\diamond}(H) = R_{\oplus}(H) = R_{\triangle}(H) = \emptyset$. Thus, the statement of the corollary is an immediate consequence of Theorem 4.9 and Lemma 4.7. □

Corollary 4.11. *There exists a polynomial-time algorithm for the problem of determining if a given line graph is localizable.*

Proof. Let G be a line graph. Since a graph H such that $G = L(H)$ can be computed in linear time [40,55], we may assume that we know H. Moreover, since G is localizable if and only if each component of G is localizable, we may assume that G (and therefore H) is connected. We may also assume that H is not isomorphic to any graph in the set $\{K_3, K_4, \text{diamond}\}$.

We apply Theorem 4.9. We compute the pendant subgraphs of H, the sets $R_{\triangle}(H)$, $R_{\diamond}(H)$, and $R_{\oplus}(H)$ of their roots, and the pendant reduction F of H. If F is not connected and bipartite, then since H is not isomorphic to K_3 or K_4, by Theorem 4.9 it follows that G is not localizable. Hence we may assume that F is a connected bipartite graph, with parts U and W. Under these assumptions, to verify whether $L(H)$ is localizable we only need to verify whether there exists a part of the bipartition of F, say U, satisfying the following conditions:

(a) $R_{\diamond}(H) \cup R_{\oplus}(H) \subseteq U$.
(b) $R_{\triangle}(H) \cap U = \emptyset$, and
(c) each vertex $u \in U \setminus R_{\oplus}(H)$ is strong in the graph $F - (N_F(u) \cap R_{\triangle}(H))$.

All the above computations as well as the verification of conditions (a) and (b) can be carried out in linear time. It only remains to justify that conditions (c) can be tested in polynomial time. Note that a vertex $u \in U \setminus R_{\oplus}(H)$ is not strong in the graph $F' = F - (N_F(u) \cap R_{\triangle}(H))$ if and only if F' has a matching M such that every neighbor of u is incident with an edge of M. The existence of such a matching can be determined by a maximum matching computation in the (bipartite) subgraph of F' induced by the vertices of distance 1 or 2 from u. Thus, condition (c) can be verified in polynomial time by carrying out $O(|U|)$ bipartite matching computations, which is well-known to be polynomially solvable (see, e.g., [35]). □

In the rest of this section, we prove Theorem 4.9. This will be done through a sequence of lemmas. Our first lemma gives a translation of the property that $G = L(H)$ is localizable to the graph H. Recall that a vertex $v \in V(H)$ is said to be *strong* if every maximal matching of H covers v. We say that a triangle T in a graph H is *strong* if every maximal matching of H contains an edge of T. A *decomposition* of a graph H is a set $\mathcal{F}$ of subgraphs of H such that each edge of H appears in exactly one subgraph from $\mathcal{F}$ (we also say that $\mathcal{F}$ *decomposes* H).

Lemma 4.12. *Let $G = L(H)$. Then, G is localizable if and only if H contains an independent set S of strong vertices and a set $\mathcal{T}$ of strong triangles decomposing $H - S$.*

Proof. Consider a graph H and suppose that its line graph, $G = L(H)$, is localizable. Thus, the vertices of G can be partitioned into $k = \alpha(G)$ strong cliques $C_1, \dots, C_k$. Note that every strong clique is maximal. Each maximal clique of G corresponds to either a triangle in H or to a star of H, that is, to a set of edges of the form $E(v) = \{e \in E(H) : v \text{ is an endpoint of } e\}$ for some $v \in V(H)$ (in which case we say that $E(v)$ is the star *centered at* v). Suppose that the cliques $C_1, \dots, C_{k'}$ correspond to triangles $T_1, \dots, T_{k'}$ in H, and the cliques $C_{k'+1}, \dots, C_k$ correspond to stars $S_{k'+1}, \dots, S_k$ of H, respectively. Since $C_1, \dots, C_k$ is a partition of $V(G)$, and $G = L(H)$, it follows that the triangles $T_1, \dots, T_{k'}$ and stars $S_{k'+1}, \dots, S_k$ form a partition of the edge set of H. Let i and j be two distinct elements of $\{k' + 1, \dots, k\}$ and let v_i, v_j be the centers of S_i and S_j, respectively, that is, $S_i = E(v_i)$ and $S_j = E(v_j)$. Since S_i and S_j are disjoint, it follows that v_i and v_j are non-adjacent. It follows that the centers of the stars $S_{k'+1}, \dots, S_k$ form an independent set of H, say S. Since for each $i \in \{1, \dots, k'\}$ and $j \in \{k' + 1, \dots, k\}$, triangle T_i is disjoint from star $S_j = E(v_j)$, the definition of $E(v_j)$ implies that v_j is not a vertex of T_i. This shows that for each $i \in \{1, \dots, k'\}$, triangle T_i is a subgraph of $H - S$. Since $\{T_1, \dots, T_{k'}, S_{k'+1}, \dots, S_k\}$ is a partition of the edge set of H, each edge of $H - S$ belongs to a unique triangle T_i. It follows that the triangles $T_1, \dots, T_{k'}$ are subgraphs of $H - S$ decomposing $H - S$. Since every maximal independent set of G intersects every strong clique $C_1, \dots, C_k$, we deduce that every maximal matching in H intersects every triangle T_i and every star S_j for $i = 1, \dots, k'$ and $j = k' + 1, \dots, k$, that is, each triangle T_i is strong and each vertex in S is strong.

Conversely, suppose that H contains an independent set S of strong vertices and a set $\mathcal{T}$ of strong triangles decomposing $H - S$. Every triangle in $\mathcal{T}$ corresponds to a clique in $G = L(H)$ and every star with center $s \in S$ also corresponds to a clique in G. This clearly gives us a partition of the vertex set of G into cliques $C_1, \dots, C_k$. Furthermore, since every maximal matching in H covers S and contains an edge from each triangle in $\mathcal{T}$, it follows that every maximal independent set of G intersects every maximal clique $C_1, \dots, C_k$, and thus each of these cliques is strong. We conclude that G is localizable. □

Given a graph H such that $L(H)$ is localizable, a pair $(S, \mathcal{T})$ such that S is an independent set of strong vertices in H and $\mathcal{T}$ is a set of strong triangles decomposing $H - S$ will be referred to as a *line-localizability certifier* of H.

The next lemma characterizes strong triangles of a graph H. A *bull* is a graph with vertices a, b, c, d, e and edges ab, bc, cd, be, ce. A triangle T in a graph H is said to be *contained in a bull* if there exists a subgraph of H isomorphic to a bull that contains T as a subgraph.

Lemma 4.13. *For every triangle T in a connected graph H of order at least 4, the following conditions are equivalent:*

1. *T is strong in H.*
2. *T is not contained in any bull.*
3. *T is either a pendant triangle, or is contained in a pendant diamond or in a pendant K_4.*

Proof. Let T be a triangle in H. Clearly, T is not strong if and only if there exists a matching M in H not containing any edge of T and covering at least two vertices of T. A minimal matching with such properties consists of two edges, each of which is incident with a vertex of T, and hence forms a bull together with T. Therefore, T is not strong if and only if T is contained in a bull.

If T is either a pendant triangle, or is contained in a pendant diamond or in a pendant K_4, then T is not contained in any bull. Conversely, suppose that T is not contained in any bull, and let U be the set of vertices in T with a neighbor outside T. Since H is connected and of order at least 4 it follows that $|U| \geq 1$. If $|U| = 1$ then T is a pendant triangle. If $|U| = 2$ then, since T is not contained in any bull, there is a unique vertex outside T with a neighbor in T, and hence T is contained in a pendant diamond. Similarly, if $|U| = 3$ then T is contained in a pendant K_4. This establishes the equivalence of the last two conditions. □

The next lemma shows that with the exception of two small cases, the strong triangles in H are pairwise edge-disjoint.

Lemma 4.14. *Let H be a connected graph that is not isomorphic to either K_4 or to the diamond. Then, any two strong triangles in H are edge-disjoint.*

Proof. Let H be a connected graph with a pair $T = \{a, b, c\}$ and $T' = \{a, b, d\}$ of strong triangles sharing an edge (namely, ab). By Lemma 4.13, each of T and T' is either a pendant triangle, or is contained in a pendant diamond or in a pendant K_4. Since each of T and T' has at most one vertex of degree 2, none of them can be a pendant triangle. Suppose first that one of them, say T, is contained in a pendant diamond. Then the vertex set of this diamond is exactly $\{a, b, c, d\}$, and the only remaining possibility for T' is that it is also contained in a pendant diamond. Since H is connected, we infer that H is isomorphic to a diamond in this case. If T is contained in a pendant K_4, then, similarly, the vertex set of this K_4 is exactly $\{a, b, c, d\}$, and the only remaining possibility for T' is that it is also contained in a pendant K_4, hence $H \cong K_4$ in this case. □

Let H be a connected graph that is not isomorphic to any graph in the set $\{K_3, K_4, \text{diamond}\}$. Recall that the *pendant reduction* of H is the graph obtained from H by deleting from H all non-root vertices of its pendant subgraphs. The next lemma establishes some necessary conditions for $L(H)$ to be localizable.

Lemma 4.15. *Let H be a connected graph not isomorphic to any graph in the set $\{K_3, K_4, \text{diamond}\}$ such that $L(H)$ is localizable and let $(S, \mathcal{T})$ be a line-localizability certifier of H. Then:*

1. *$\mathcal{T}$ is the set of all strong triangles of H.*
2. *$R_{\triangle}(H) \cap S = \emptyset$.*
3. *$R_{\diamond}(H) \cup R_{\oplus}(H) \subseteq S$.*
4. *The pendant reduction F of H is a connected bipartite graph with a bipartition of its vertex set $\{S, V(F) \setminus S\}$.*

Proof. Let H, S, and $\mathcal{T}$ be as in the statement of the lemma. Let $\mathcal{T}_{\triangle}$ denote the set of pendant triangles of H, and let $\mathcal{T}_{\diamond}$ and $\mathcal{T}_{\oplus}$ denote the sets of triangles of H contained in a pendant diamond or in a pendant K_4, respectively. By Lemma 4.13, each strong triangle of H belongs to one (and then to exactly one) of the sets $\mathcal{T}_{\triangle}$, $\mathcal{T}_{\diamond}$, and $\mathcal{T}_{\oplus}$. Moreover, by Lemma 4.14, any two triangles in $\mathcal{T}_{\triangle} \cup \mathcal{T}_{\diamond} \cup \mathcal{T}_{\oplus}$ are pairwise edge-disjoint.

By Lemma 4.13, in order to show statement 1, we need to show that $\mathcal{T} = \mathcal{T}_{\triangle} \cup \mathcal{T}_{\diamond} \cup \mathcal{T}_{\oplus}$. Since every triangle in $\mathcal{T}$ is strong, we have $\mathcal{T} \subseteq \mathcal{T}_{\triangle} \cup \mathcal{T}_{\diamond} \cup \mathcal{T}_{\oplus}$. We prove the converse inclusion $\mathcal{T}_{\triangle} \cup \mathcal{T}_{\diamond} \cup \mathcal{T}_{\oplus} \subseteq \mathcal{T}$ in three steps. Along the way we will also prove statements 2 and 3 above.

First, let $T = \{a, b, c\} \in \mathcal{T}_{\triangle}$ be a pendant triangle with root c. Note that none of the vertices a and b is strong. Therefore, $\{a, b\} \cap S = \emptyset$ and consequently there is a strong triangle $T' \in \mathcal{T}$ such that $\{a, b\} \subseteq T'$. Since T is the only triangle of H containing the edge ab, we infer that $T' = T$ and hence $T \in \mathcal{T}$. This shows that $\mathcal{T}_{\triangle} \subseteq \mathcal{T}$. Moreover, since $T \in \mathcal{T}$, we have $T \cap S = \emptyset$ and consequently $c \notin S$. This shows statement 2, that is, that no root of a pendant triangle is in S.

Second, let $T = \{a, b, c\} \in \mathcal{T}_{\diamond}$ be a triangle contained in a pendant diamond with root d such that $d_H(a) = 2$, $d_H(b) = d_H(c) = 3$. Since vertex a is not strong, we have $a \notin S$. Since S is independent, one of b and c, say b, does not belong to S. It follows that the edge ab is contained in some triangle $T' \in \mathcal{T}$. Again, since T is the only triangle containing the edge ab, we infer that $T' = T$ and hence $T \in \mathcal{T}$. This shows that $\mathcal{T}_{\diamond} \subseteq \mathcal{T}$. By Lemma 4.14, the strong triangles of H are pairwise edge-disjoint, therefore the triangle $\{b, c, d\}$ is not strong. It follows that $d \in S$ since otherwise the edge bd would be an edge of $H - S$ not covered by any triangle in $\mathcal{T}$.

Now, let $T = \{a, b, c\} \in \mathcal{T}_{\oplus}$ be a triangle contained in a pendant K_4 with root d and $d_H(a) = d_H(b) = d_H(c) = 3$. Suppose for a contradiction that $T \notin \mathcal{T}$. Since S is independent, it contains at most one vertex of T. We may thus assume that $\{a, b\} \cap S = \emptyset$. Let T' be the triangle in $\mathcal{T}$ covering the edge ab. Since $T \notin \mathcal{T}$, we infer that $T' \neq T$ and therefore $T' = \{a, b, d\}$. Since the strong triangles of H are pairwise edge-disjoint and T is a strong triangle sharing the edge ab with T', we infer that T' is not strong, contrary to the fact that $T' \in \mathcal{T}$. This shows that $\mathcal{T}_{\oplus} \subseteq \mathcal{T}$. Similarly as above, using the fact that the strong triangles of H are pairwise edge-disjoint, we infer that in order to cover the edge ad, we must have $d \in S$. The above two paragraphs show statement 3, that is, that the root of every pendant diamond or K_4 is in S.

Finally, we show statement 4. Let F be the pendant reduction of H. Since H is connected and F differs from H only by non-root vertices of its pendant subgraphs, we infer that F is connected. Since $\mathcal{T}$ is the set of all strong triangles of H and $T \cap S = \emptyset$ for all $T \in \mathcal{T}$, it follows that $S \subseteq V(F)$; in particular, S is an independent set of F. It thus suffices to show that the set $V(F) \setminus S$ is independent. We have $V(F) = Z \cup R_{\triangle}(H)$, where Z is the set of vertices of H not contained in any strong triangle. Suppose for a contradiction that there is an edge $uv \in E(H)$ connecting two vertices of $V(F) \setminus S$. Then uv is an edge of the graph $H - S$. If $\{u, v\} \cap Z \neq \emptyset$, then uv would be an edge of $H - S$ not contained in any triangle in $\mathcal{T}$, contrary to the assumption on $\mathcal{T}$. Thus, we have $\{u, v\} \subseteq R_{\triangle}$, therefore u and v are adjacent roots of two pendant triangles. Since uv is an edge of $H - S$, it is contained in some triangle $T \in \mathcal{T}$, which is clearly impossible due to the characterization of strong triangles given by Lemma 4.13. This contradiction proves statement 4. □

Now we have everything ready to prove the announced characterization of localizable line graphs, which we restate here for convenience.

Theorem 4.9 (*Restated*). *Let H be a connected graph and let $G = L(H)$. Then, G is localizable if and only if one the following holds:*

1. *H is not isomorphic to any graph in the set $\{K_3, K_4, \text{diamond}\}$ and its pendant reduction F is a connected bipartite graph with a bipartition of its vertex set into two independent sets U and W such that*
   (a) $R_{\Diamond}(H) \cup R_{\oplus}(H) \subseteq U$,
   (b) $R_{\triangle}(H) \subseteq W$, *and*
   (c) *each vertex $u \in U \setminus R_{\oplus}(H)$ is strong in the graph $F - (N_F(u) \cap R_{\triangle}(H))$.*
2. *H is isomorphic to either K_3 or K_4.*

Proof. Suppose first that $G = L(H)$ is localizable, and let $(S, \mathcal{T})$ be a line-localizability certifier of H. Suppose that H is not isomorphic to any of K_3 or K_4. Since the line graph of the diamond is not localizable, H is also not isomorphic to the diamond. Let F be the pendant reduction of H. By construction, we have $R_{\triangle}(H) \cup R_{\Diamond}(H) \cup R_{\oplus}(H) \subseteq V(F)$. By Lemma 4.15, F is a connected bipartite graph with a bipartition of its vertex set $\{S, V(F) \setminus S\}$. The same lemma implies that $R_{\Diamond}(H) \cup R_{\oplus}(H) \subseteq S$ and $R_{\triangle}(H) \subseteq V(F) \setminus S$. Thus, letting $W' = N_F(u) \cap R_{\triangle}(H)$ and $F' = F - W'$, we only need to show that each vertex $u \in S \setminus R_{\oplus}(H)$ is strong in graph F', and the desired conclusion will follow by taking $U = S$ and $W = V(F) \setminus S$.

Suppose for a contradiction that there exists a vertex $u \in S \setminus R_{\oplus}(H)$ that is not strong in graph F'. Then there exists a maximal matching M' of F' such that u is not incident with any edge of M'. Since $u \in S$, vertex u is strong in H, that is, every maximal matching of H covers u. However, we will now show that M' is contained in a maximal matching of H not covering u. Since $u \in S$ and $R_{\triangle}(H) \subseteq V(F) \setminus S$, vertex u is not the root of any pendant triangle of H. Moreover, since $u \notin R_{\oplus}(H)$, it is also not the root of any pendant K_4. Let $\mathcal{D}$ be the (possibly empty) set of pendant diamonds in H having u as the root. For each pendant diamond $D \in \mathcal{D}$, let e_D be the edge of D completing a triangle with u. For each vertex $w \in W'$, let $\mathcal{T}_w = \{T_1, \dots, T_k\}$ be the set of pendant triangles of H with root w, let $T_i = \{w, a_i, b_i\}$, and let $M_w = \{wa_1\} \cup \{a_ib_i : 2 \leq i \leq k\}$. Note that M_w is a matching in H. Let

$$M = M' \cup \{e_D : D \in \mathcal{D}\} \cup \bigcup_{w \in W'} M_w \,.$$

Then, M is a matching in H covering all neighbors of u. Therefore, M' is contained in a maximal matching of H not covering u, contradicting the fact that u is strong in H. This completes the proof of the forward direction.

For the converse direction, suppose that one of conditions 1 and 2 in the statement of the theorem holds. If H is isomorphic to one of K_3 and K_4, then its line graph is localizable. Suppose now that condition 1 holds, that is, H is not isomorphic to any graph in the set $\{K_3, K_4, \text{diamond}\}$ and its pendant reduction F is a connected bipartite graph with a bipartition of its vertex set into two independent sets U and W such that conditions (a)–(c) hold. Let $\mathcal{T}$ be the set of all strong triangles of H. We will show that the pair $(U, \mathcal{T})$ is a line-localizability certifier of H, which will imply that G is localizable by Lemma 4.12. In other words, we need to show that U is an independent set of strong vertices in H and that $\mathcal{T}$ is a set of strong triangles decomposing $H - U$. By Lemma 4.13, $\mathcal{T}$ equals the set containing all pendant triangles, all triangles contained in a pendant diamond, and all triangles contained in a pendant K_4. Note that by Lemma 4.14, any two strong triangles in H are edge-disjoint. Therefore, the fact that F is bipartite and properties (a) and (b) imply that $\mathcal{T}$ decomposes $H - U$. Clearly, U is an independent set in H. It therefore remains to show that every vertex $u \in U$ is strong in H. Suppose for a contradiction that some $u \in U$ is not strong in H. Then, there exists a maximal matching M of H not covering u. We consider two cases. Suppose first that $u \in U \setminus R_{\oplus}(H)$. By property (c), vertex u is strong in the graph $F' = F - (N_F(u) \cap R_{\triangle}(H))$. It follows that matching $M' = M \cap E(F')$ covers all vertices in $N_F(u) \setminus R_{\triangle}(H)$, which are exactly the neighbors of u in F'. Extending M' to a maximal matching of F' not covering u shows that u is not strong in F', a contradiction. Suppose now that $u \in R_{\oplus}(H)$. Let D be a pendant K_4 of H with vertex set $\{a, b, c, u\}$ and with root u. Since u is not covered by M, matching M covers at most two of the vertices in $\{a, b, c\}$. Adding to M the edge connecting u with an uncovered vertex in $\{a, b, c\}$ results in a matching of H properly containing M, which contradicts the maximality of M. This completes the proof. $\square$

5. Concluding remarks

In this paper we have initiated a study of localizable graphs, a rich subclass of well-covered graphs. The two properties coincide in the class of perfect graphs, and more generally in any class of graphs in which the clique cover number equals the independence number. We gave efficiently testable characterizations of localizable graphs within the classes of triangle-free graphs, C_4-free graphs, and line graphs. Based on properties of localizable graphs, we disproved a conjecture due to Zaare-Nahandi about k-partite well-covered graphs having all maximal cliques of size k.

Our work leaves open many questions related to localizable graphs. For example, is it polynomial to check whether a given planar graph is localizable? (The corresponding question for well-covered graphs was asked in 1994 by Dean and Zito [18] and seems to be open.) Testing localizability is polynomial for triangle-free graphs. What is the complexity of the problem for K_4-free graphs (and, more generally, for graphs of bounded clique number)? Is it polynomial to check whether a given comparability graph is localizable? (Equivalently, given a partially ordered set, is it polynomial to test whether all its maximal antichains are of the same size?) It follows from Theorem 3.3 that it is NP-hard to recognize graphs such that both G and its complement are localizable. What is the complexity of recognizing well-covered co-well-covered graphs?

Note Added in Proof. While this paper was in the reviewing process, Habib, Milanič, and Mydlarz answered one of the above questions. They showed that the problem of determining whether a given comparability graph is localizable is solvable in polynomial time.

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References

[1] N. Alon, A. Kostochka, B. Reiniger, D.B. West, X. Zhu, Coloring, sparseness and girth, Israel J. Math. 214 (1) (2016) 315–331.
[2] B. Benedetti, M. Varbaro, Unmixed graphs that are domains, Comm. Algebra 39 (7) (2011) 2260–2267.
[3] C. Berge, Stochastic graphs and strongly perfect graphs—a survey, Southeast Asian Bull. Math. 7 (1) (1983) 16–25.
[4] C. Berge, P. Duchet, Strongly perfect graphs, in: Topics on Perfect Graphs, in: North-Holland Math. Stud., vol. 88, North-Holland, Amsterdam, 1984, pp. 57–61.
[5] E. Boros, N. Chiarelli, M. Milanič, Equistarable bipartite graphs, Discrete Math. 339 (7) (2016) 1960–1969.
[6] E. Boros, V. Gurvich, M. Milanič, On CIS circulants, Discrete Math. 318 (2014) 78–95.
[7] E. Boros, V. Gurvich, M. Milanič, On equistable, split, CIS, and related classes of graphs, Discrete Appl. Math. 216 (part 1) (2017) 47–66.
[8] A. Brandstädt, V.B. Le, J.P. Spinrad, Graph classes: A Survey, in: SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999, p. xii+304.
[9] J.I. Brown, R.J. Nowakowski, I.E. Zverovich, The structure of well-covered graphs with no cycles of length 4, Discrete Math. 307 (17–18) (2007) 2235–2245.
[10] M. Burlet, J. Fonlupt, Polynomial algorithm to recognize a Meyniel graph, in: Topics on Perfect Graphs, in: North-Holland Math. Stud., vol. 88, North-Holland, Amsterdam, 1984, pp. 225–252.
[11] S.R. Campbell, M.N. Ellingham, G.F. Royle, A characterisation of well-covered cubic graphs, J. Combin. Math. Combin. Comput. 13 (1993) 193–212.
[12] Y. Caro, M.N. Ellingham, J.E. Ramey, Local structure when all maximal independent sets have equal weight, SIAM J. Discrete Math. 11 (4) (1998) 644–654.
[13] I.D. Castrillón, R. Cruz, E. Reyes, On well-covered, vertex decomposable and Cohen-Macaulay graphs, Electron. J. Combin. 23 (2) (2016) 17 Paper 2.39.
[14] M.-S. Chang, Efficient algorithms for the domination problems on interval and circular-arc graphs, SIAM J. Comput. 27 (6) (1998) 1671–1694 (electronic).
[15] M.-S. Chang, S.-y. Hsieh, G.-H. Chen, Dynamic programming on distance-hereditary graphs, in: Algorithms and Computation (Singapore, 1997), in: Lecture Notes in Comput. Sci., vol. 1350, Springer, Berlin, 1997, pp. 344–353.
[16] V. Chvátal, P.J. Slater, A note on well-covered graphs, in: Quo vadis, graph theory?, in: Ann. Discrete Math., vol. 55, North-Holland, Amsterdam, 1993, pp. 179–181.
[17] B. Courcelle, J.A. Makowsky, U. Rotics, Linear time solvable optimization problems on graphs of bounded clique-width, Theory Comput. Syst. 33 (2) (2000) 125–150.
[18] N. Dean, J. Zito, Well-covered graphs and extendability, Discrete Math. 126 (1–3) (1994) 67–80.
[19] E. Dobson, A. Hujdurović, M. Milanič, G. Verret, Vertex-transitive CIS graphs, European J. Combin. 44 (part A) (2015) 87–98.
[20] J. Edmonds, Paths, trees, and flowers, Canad. J. Math. 17 (1965) 449–467.
[21] O. Favaron, Very well covered graphs, Discrete Math. 42 (2–3) (1982) 177–187.
[22] A. Finbow, B. Hartnell, A characterization of parity graphs containing no cycle of order five or less, Ars Combin. 40 (1995) 227–234.
[23] A. Finbow, B. Hartnell, R.J. Nowakowski, A characterization of well covered graphs of girth 5 or greater, J. Combin. Theory Ser. B 57 (1) (1993) 44–68.
[24] A.S. Finbow, B.L. Hartnell, A game related to covering by stars, Ars Combin. 16 (A) (1983) 189–198.
[25] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, second ed., in: Annals of Discrete Mathematics, vol. 57, Elsevier Science B.V., Amsterdam, 2004, p. xxvi+314.
[26] M.C. Golumbic, U. Rotics, On the clique-width of some perfect graph classes, Internat. J. Found. Comput. Sci. 11 (3) (2000) 423–443.
[27] M. Grötschel, L. Lovász, A. Schrijver, Geometric algorithms and Combinatorial Optimization, in: Algorithms and Combinatorics: Study and Research Texts, vol. 2, Springer-Verlag, Berlin, 1988, p. xii+362.
[28] B. Grünbaum, Matchings in polytopal graphs, Networks 4 (1974) 175–190.
[29] P. Hall, On representatives of subsets, J. Lond. Math. Soc. s1-10 (1) (1935) 26–30.
[30] B.L. Hartnell, Well-covered graphs, J. Combin. Math. Combin. Comput. 29 (1999) 107–115.
[31] R.B. Hayward, Weakly triangulated graphs, J. Combin. Theory Ser. B 39 (3) (1985) 200–208.
[32] J. Herzog, T. Hibi, X. Zheng, Cohen-Macaulay chordal graphs, J. Combin. Theory Ser. A 113 (5) (2006) 911–916.
[33] C.T. Hoàng, On a conjecture of Meyniel, J. Combin. Theory Ser. B 42 (3) (1987) 302–312.
[34] C.T. Hoàng, Efficient algorithms for minimum weighted colouring of some classes of perfect graphs, Discrete Appl. Math. 55 (2) (1994) 133–143.
[35] J.E. Hopcroft, R.M. Karp, An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs, SIAM J. Comput. 2 (1973) 225–231.
[36] W.L. Hsu, K.-H. Tsai, Linear time algorithms on circular-arc graphs, Inform. Process. Lett. 40 (3) (1991) 123–129.
[37] M. Kamiński, V. Lozin, Coloring edges and vertices of graphs without short or long cycles, Contrib. Discrete Math. 2 (1) (2007) 61–66.
[38] T. Kloks, C.-M. Lee, J. Liu, H. Müller, On the recognition of general partition graphs, in: Graph-Theoretic Concepts in Computer Science, in: Lecture Notes in Comput. Sci., 2880, Springer, Berlin, 2003, pp. 273–283.
[39] D. Kratsch, L. Stewart, Domination of cocomparability graphs, SIAM J. Discrete Math. 6 (3) (1993) 400–417.
[40] P.G.H. Lehot, An optimal algorithm to detect a line graph and output its root graph, J. Assoc. Comput. Mach. 21 (1974) 569–575.
[41] M. Lesk, M.D. Plummer, W.R. Pulleyblank, Equi-matchable graphs, in: Graph Theory and Combinatorics (Cambridge, 1983), Academic Press, London, 1984, pp. 239–254.
[42] M. Lewin, Matching-perfect and cover-perfect graphs, Israel J. Math. 18 (1974) 345–347.
[43] L. Lovász, Normal hypergraphs and the perfect graph conjecture, Discrete Math. 2 (3) (1972) 253–267.

[44] K. McAvaney, J. Robertson, D. DeTemple, A characterization and hereditary properties for partition graphs, Discrete Math. 113 (1–3) (1993) 131–142.
[45] H. Meyniel, On the perfect graph conjecture, Discrete Math. 16 (4) (1976) 339–342.
[46] M. Milanič, N. Trotignon, Equistarable graphs and counterexamples to three conjectures on equistable graphs, J. Graph Theory 84 (2017) 536–551.
[47] S. Morey, R.H. Villarreal, Edge ideals: algebraic and combinatorial properties, in: Progress in Commutative Algebra 1, de Gruyter, Berlin, 2012, pp. 85–126.
[48] J. Mycielski, Sur le coloriage des graphs, Colloq. Math. 3 (1955) 161–162.
[49] S.-i. Oum, P. Seymour, Approximating clique-width and branch-width, J. Combin. Theory Ser. B 96 (4) (2006) 514–528.
[50] M.D. Plummer, Some covering concepts in graphs, J. Comb. Theory 8 (1970) 91–98.
[51] M.D. Plummer, Well-covered graphs: a survey, Quaestiones Math. 16 (3) (1993) 253–287.
[52] E. Prisner, J. Topp, P.D. Vestergaard, Well covered simplicial, chordal, and circular arc graphs, J. Graph Theory 21 (2) (1996) 113–119.
[53] M. Rao, MSOL partitioning problems on graphs of bounded treewidth and clique-width, Theoret. Comput. Sci. 377 (1–3) (2007) 260–267.
[54] G. Ravindra, Well-covered graphs, J. Combinatorics Information Syst. Sci. 2 (1) (1977) 20–21.
[55] N.D. Roussopoulos, A max $\{m, n\}$ algorithm for determining the graph H from its line graph G, Inform. Process. Lett. 2 (1973) 108–112.
[56] R.S. Sankaranarayana, L.K. Stewart, Complexity results for well-covered graphs, Networks 22 (3) (1992) 247–262.
[57] A. Schrijver, Combinatorial optimization. Polyhedra and Efficiency, in: Algorithms and Combinatorics, vol. 24, Springer-Verlag, Berlin, 2003.
[58] J.A. Staples, On some subclasses of well-covered graphs, (Ph.D. thesis), Vanderbilt University, Department of Mathematics, 1975.
[59] D. Tankus, M. Tarsi, Well-covered claw-free graphs, J. Combin. Theory Ser. B 66 (2) (1996) 293–302.
[60] D. Tankus, M. Tarsi, The structure of well-covered graphs and the complexity of their recognition problems, J. Combin. Theory Ser. B 69 (2) (1997) 230–233.
[61] R.H. Villarreal, Cohen-macaulay graphs, Manuscripta Math. 66 (3) (1990) 277–293.
[62] D.B. West, Introduction to Graph Theory, Prentice Hall, Inc., Upper Saddle River, NJ, 1996, p. xvi+512.
[63] M. Yamashita, T. Kameda, Modeling k-coteries by well-covered graphs, Networks 34 (3) (1999) 221–228.
[64] R. Zaare-Nahandi, Pure simplicial complexes and well-covered graphs, Rocky Mountain J. Math. 45 (2) (2015) 695–702.
[65] W. Zang, Generalizations of Grillet's theorem on maximal stable sets and maximal cliques in graphs, Discrete Math. 143 (1–3) (1995) 259–268.
[66] A.A. Zykov, On some properties of linear complexes, Amer. Math. Soc. Translation 1952 (79) (1952) 33.