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Blockers and transversals

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ABSTRACT

Given an undirected graph $G = (V, E)$ with matching number $\nu(G)$, we define d-blockers as subsets of edges B such that $\nu((V, E \setminus B)) \leq \nu(G) - d$. We define d-transversals T as subsets of edges such that every maximum matching M has $|M \cap T| \geq d$. We explore connections between d-blockers and d-transversals. Special classes of graphs are examined which include complete graphs, regular bipartite graphs, chains and cycles and we construct minimum d-transversals and d-blockers in these special graphs. We also study the complexity status of finding minimum transversals and blockers in arbitrary graphs.

1. Introduction

In this paper we introduce the following two concepts: in an undirected graph $G = (V, E)$ a set of edges T such that each maximum matching in G contains at least a given number d of edges of T is a d-transversal; a d-blocker is a set of edges B such that the matching number (the cardinality of a maximum matching) of $(V, E \setminus B)$ is at most the matching number of G minus d. We will consider the problem of finding a minimum d-transversal T and a minimum d-blocker B in G.

The problem of the d-blocker is closely related to some edge deletion and edge modification problems which have been studied in [4,13,14]. Similar problems have also been analyzed for vertices (see [5,12,15]).

In [10,11], the authors consider the problem of existence of a maximum matching whose removal leads to a graph with given upper (resp. lower) bound for the cardinality of its maximum matching. Here we will not impose any structure on the edge set representing the d-blocker.

In [2] a minimal blocker for a bipartite graph G is defined as a minimal set of edges the removal of which leaves no perfect matching in G and explicit characterizations of minimal blockers of bipartite graphs are given. An efficient algorithm enumerating the minimal blockers is given.

A concept close to d-transversal can be found in [3] where authors consider the notion of multiple transversal, another generalization of a transversal in the hypergraph of perfect matchings. Here a multiple transversal must intersect each perfect matching M_i with at least b_i edges.

A different concept of d-transversals has been studied in [6]. Given a set of integers $\{p_0, p_1, \ldots, p_s\}$ and a bipartite graph G, one has to find a minimum set of edges R such that for each p_i, $i = 0, 1, \ldots, s$, there exists a maximum matching M_i with $|M_i \cap R| = p_i$. Results have been given for special classes of bipartite graphs.

For some applications of the concept of colored blockers and transversals we refer the reader to [8].

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Our paper is organized as follows. In Section 2, we give some definitions and show some basic properties concerning transversals and blockers. We also study the connections between both notions. Section 3 deals with complexity results. We show that given two integers d and k, deciding whether there exists a d-transversal or a d-blocker of size k is $\mathcal{NP}$-complete in bipartite graphs. Some special classes of graphs are analyzed in Section 4. These include complete graphs, regular bipartite graphs, chains and cycles.

2. Definitions and basic properties

All graph theoretical terms not defined here can be found in [1]. Throughout this paper we are concerned with undirected simple loopless graphs $G = (V, E)$. The degree of a vertex v is denoted $d(v)$ and $\Delta(G)$ stands for the maximum degree of a vertex in G. G will be assumed connected. A **cut-edge** $e = uv$ is an edge such that its removal disconnects G. A **matching** M is a set of pairwise non-adjacent edges. A matching M is called **maximum** if its cardinality $|M|$ is maximum. The largest cardinality of a matching in G, its **matching number**, will be denoted by $\nu(G)$. More specifically we will be interested in subsets of edges which will intersect maximum matchings in G or whose removal will reduce by a given number the matching number.

We shall say that a subset $T \subseteq E$ is a ***d*-transversal** of G if for every maximum matching $M \in G$ we have $|M \cap T| \geq d$. Thus a d-transversal is a subset of edges which intersect each maximum matching in at least d edges.

A subset $B \subseteq E$ will be called a ***d*-blocker** of G if $\nu(G') \leq \nu(G) - d$ where G' is the partial graph $G' = (V, E \setminus B)$. So B is a subset of edges such that its removal reduces by at least d the cardinality of a maximum matching.

In case where $d = 1$, a d-transversal or a d-blocker is called a **transversal** or a **blocker**, respectively. We remark that in this case our definition of a transversal coincides with the definition of a transversal in the hypergraph of maximum matchings of G.

We denote by $\boldsymbol{\beta_d(G)}$ the minimum cardinality of a d-blocker in G and by $\boldsymbol{\tau_d(G)}$ the minimum cardinality of a d-transversal in G ($\beta(G)$ and $\tau(G)$ in case of a blocker or a transversal).

Let v be a vertex in graph G. The **bundle** of v, denoted by $\omega(v)$, is the set of edges which are incident to v. So $|\omega(v)| = d(v)$ is the degree of v. As we will see, bundles play an important role in finding d-transversals and d-blockers.

Let $\boldsymbol{P_0(G)} = \{vw \in E \mid \forall$ maximum matching $M, vw \notin M\}$ and $\boldsymbol{P_1(G)} = \{vw \in E \mid \forall$ maximum matching $M, vw \in M\}$. Let M be a matching. A vertex $v \in V$ is called **saturated by** M if there exists an edge $vw \in M$. A vertex $v \in V$ is called **strongly saturated** if for all maximum matchings M, v is saturated by M. We denote by $\boldsymbol{S(G)}$ the set of strongly saturated vertices of a graph G.

Notice that the sets $P_0(G)$, $P_1(G)$ and $S(G)$ can be determined in polynomial time. In fact, if we want to test whether an edge vw belongs to $P_0(G)$, we delete all edges having exactly one endpoint in $\{v, w\}$ and we determine a maximum matching M in the remaining graph. Then vw belongs to $P_0(G)$ if and only if $|M| = \nu(G) - 1$. In order to check whether an edge vw is in $P_1(G)$, we simply delete this edge and find a maximum matching M in the remaining graph. Then vw belongs to $P_1(G)$ if and only if $|M| = \nu(G) - 1$. By performing these tests for all edges in G, we determine the sets $P_0(G)$ and $P_1(G)$. Since a maximum matching in a graph can be found in polynomial time (see [7]), $P_0(G)$ and $P_1(G)$ can be determined in polynomial time. Concerning $S(G)$, first notice that all vertices which are incident to an edge of $P_1(G)$ necessarily belong to $S(G)$. For each other vertex v, to check whether it is strongly saturated, we simply delete it in G and find a maximum matching M in the remaining graph. Then v must belong to $S(G)$ if and only if $|M| = \nu(G) - 1$.

Remark 2.1. If G is a graph such that $|P_1(G)| \geq d$, a minimum d-transversal is obtained by taking d edges in $P_1(G)$. This is not necessarily true for a minimum d-blocker. In fact, consider the chain $C = \{x_1x_2, x_2x_3, x_3x_4\}$. We have $P_1(C) = \{x_1x_2, x_3x_4\}$, but clearly $P_1(C)$ is a blocker but not a 2-blocker for C.

We will now give some basic properties concerning d-transversals and d-blockers in a graph $G = (V, E)$. We shall always assume that $d \leq \nu(G)$.

Property 2.1. *In any graph G and for any $d \geq 1$, a d-blocker B is a d-transversal.*

Proof. If the removal of $B \subseteq E$ reduces the maximum cardinality of a matching by at least d, then every maximum matching will contain at least d edges of B: indeed if there were a maximum matching M in G with $|M \cap B| \leq d - 1$, then the matching $M \setminus B$ in $(V, E \setminus B)$ has cardinality $|M \setminus B| > \nu(G) - d$, contradicting the assumption that B is a d-blocker. □

Property 2.2. *In any graph $G = (V, E)$ a set T is a transversal if and only if it is a blocker.*

Proof. From Property 2.1, we just have to show that a transversal T is a blocker. By definition we have $M \cap T \neq \emptyset$ for every maximum matching M. It follows that after the removal of T, the matching number in G has decreased by at least one. □

Observe that in any graph G and for any $d \geq 1$, a d-transversal T is a blocker. In fact, a d-transversal T is also a transversal and hence from Property 2.2 we conclude that T is a blocker.

Remark 2.2. For $d \geq 2$, there are d-transversals which are not d-blockers. Fig. 1 shows in a graph $G = C_6$ (cycle on six vertices) a set $T \subseteq E$ (bold edges) which is a 2-transversal ($|M \cap T| \geq 2$ for every maximum matching). It is not a 2-blocker, since $\nu(G) = 3$ and in $G' = (V, E \setminus T)$ we have $\nu(G') = 2 > \nu(G) - 2 = 1$.

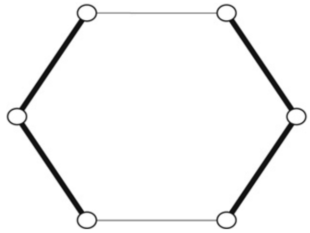

Fig. 1. A 2-transversal which is not a 2-blocker.

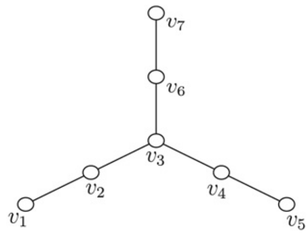

Fig. 2. Graph for which the union of 2 minimum transversals is not a minimum 2-transversal.

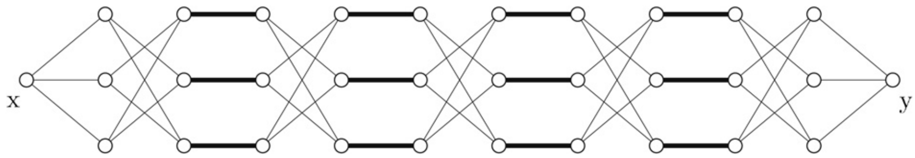

Fig. 3. A 3-regular bipartite graph with a minimum 4-transversal (bold edges), which is also a blocker but not a 2-blocker.

Property 2.3. *Let G be a graph. For any independent set $\{v_1, v_2, \dots, v_d\} \subseteq S(G)$, the set $T = \cup_{i=1}^{d} \omega(v_i) \subseteq E$, is a d-transversal.*

Proof. Since $v_i \in S(G)$ for all $i = 1, \dots, d$, any maximum matching M in G satisfies $|M \cap \omega(v_i)| = 1$ for all $i = 1, \dots, d$. As $\{v_1, v_2, \dots, v_d\}$ is an independent set in G we thus have $|M \cap T| = d$. $\square$

Notice that this is not necessarily the case for a d-blocker B. In fact, as shown in Fig. 1, the two bundles do not form a 2-blocker.

Furthermore observe that if T_1 is a d_1-transversal of a graph $G = (V, E)$ and if T_2 is a d_2-transversal of G disjoint from T_1, then $T = T_1 \cup T_2$ is clearly a $(d_1 + d_2)$-transversal of G. Nevertheless if T_1 and T_2 are minimum, then T is not necessarily minimum. This can easily be seen on the graph $G = (V, E)$ with $V = \{v_1, \dots, v_7\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_3v_6, v_6v_7\}$ (see Fig. 2). In this case two minimum disjoint transversals are $T_1 = \{v_1v_2, v_2v_3\}$ and $T_2 = \{v_3v_4, v_4v_5\}$, but the unique minimum 2-transversal is $T = \{v_1v_2, v_4v_5, v_6v_7\}$.

Now consider the chain on vertices v_1, v_2, v_3, v_4. Two minimum disjoint blockers are given by $B_1 = \{v_1v_2\}$ and $B_2 = \{v_3v_4\}$, but $B_1 \cup B_2$ is not a 2-blocker.

Property 2.4. *If T is a minimum d-transversal in a graph $G = (V, E)$ and $v_iv_j \in T$, then there exists a maximum matching M containing the edge v_iv_j and such that $|M \cap T| = d$.*

Proof. Suppose that for all maximum matchings M of G containing v_iv_j we have $|M \cap T| > d$. Then $T' = T \setminus \{v_iv_j\}$ is a d-transversal of G. This contradicts the fact that T is minimum. $\square$

Remark 2.3. For any $d \geq 2$, there exists a Δ-regular bipartite graph which admits a subset $T \subseteq E$ which satisfies the following:

1. T is a d-transversal;
2. T is a blocker but not a 2-blocker;
3. $|T| = d\Delta$ and T has minimum cardinality;
4. T is a matching.

Fig. 3 illustrates the construction for $\Delta = 3$ and $d = 4$.

One should observe that d-transversals are not necessarily formed by sets of mutually adjacent edges like bundles. We may indeed have d-transversals formed by sets of mutually non-adjacent edges like matchings.

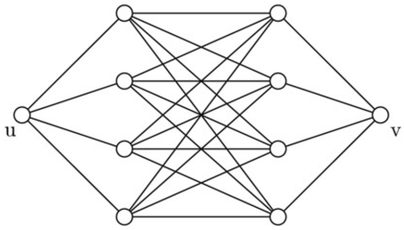

Fig. 4. A 3-gadget between u and v.

The following result will be useful for characterizing d-transversals and d-blockers in graphs having cut-edges. It can be applied for instance in enumeration schemes. It may in particular be used for dealing with trees by a dynamic programming procedure but this goes beyond the scope of this paper.

Theorem 2.1. *Let $G = (V, E)$ be a graph with $P_0(G) = P_1(G) = \emptyset$ and let vw be a cut-edge. Then exactly one of v and w is in $S(G)$.*

Proof. First suppose that $v, w \notin S(G)$. Then there are maximum matchings M, M' such that M (resp. M') does not saturate v (resp. w). Clearly M (resp. M') must saturate w (resp. v), otherwise M (resp. M') would not be a maximum matching. Let E_v (resp. E_w) be the edge set of the component of $(V, E \setminus \{vw\})$ containing v (resp. w). Let $M_v = M \cap E_v$ and $M_w = M \cap E_w$; let also $M'_v = M' \cap E_v$ and $M'_w = M' \cap E_w$. We have $|M_v| + |M_w| = \nu(G) = |M'_v| + |M'_w|$. Now $M_v \cup M'_w \cup \{vw\}$ is a matching; so is $M'_v \cup M_w$. By summing their sizes we have $|M_v| + |M'_v| + |\{vw\}| + |M'_v| + |M_w| > 2\nu(G)$ which is impossible.

Suppose now that $v, w \in S(G)$. Since $P_0(G) = P_1(G) = \emptyset$, there is a maximum matching M with $vw \notin M$ and a maximum matching M' with $vw \in M'$. Again let $M_v = M \cap E_v$, $M_w = M \cap E_w$, $M'_v = M' \cap E_v$ and $M'_w = M' \cap E_w$. We have $|M_v| + |M_w| = \nu(G) = |M'_v| + |M'_w| + 1$; w.l.o.g. we may assume $|M'_w| < |M_w|$, so $|M'_v| \geq |M_v|$. But then $M'_v \cup M_w$ is a matching with $|M'_v| + |M_w| \geq |M_v| + |M_w|$ not saturating v, which is a contradiction. $\square$

Since in a tree each edge is a cut-edge, we deduce the following corollary.

Corollary 2.2. *Let $G = (V, E)$ be a tree with $P_0(G) = P_1(G) = \emptyset$. Then for each edge vw exactly one of v and w is in $S(G)$.*

3. Complexity results

We shall now discuss the complexity of the two basic existence problems for d-blockers and d-transversals.

BLOCK(G, d, k)

Instance: An undirected graph $G = (V, E)$ and two positive integers $0 \leq d \leq \nu(G)$, $0 \leq k \leq |E|$.

Question: Does there exist a set $B \subseteq E$ with $|B| \leq k$ such that $\nu(G') \leq \nu(G) - d$ where $G' = (V, E \setminus B)$?

TRANS(G, d, k)

Instance: An undirected graph $G = (V, E)$ and two positive integers $0 \leq d \leq \nu(G)$, $0 \leq k \leq |E|$.

Question: Does there exist a set $T \subseteq E$ with $|T| \leq k$ such that for each maximum matching M in G, $|M \cap T| \geq d$?

We could also consider the problem of finding a d-blocker B (resp. d-transversal T) of size at most k in a graph $G = (V, E)$ with the additional constraint that for some given subset of edges $U \subseteq E$, we impose $B \cap U = \emptyset$ (resp. $T \cap U = \emptyset$). This problem can be polynomially reduced to *BLOCK*(G', d, k) (resp. *TRANS*(G', d, k)) where $G' = (V, E')$ is the graph obtained from G by adding for each edge $e \in U$, k edges parallel to e. This can be seen by the following observation. Let U' be the set containing all edges of U and all added edges, i.e., $U' = U \cup (E' \setminus E)$. Since each edge $e \in U$ has k parallel edges $e_1, e_2, \ldots, e_k \in E'$, there exists for any d-blocker B (resp. d-transversal T) in G' with $|B| \leq k$ (resp. $|T| \leq k$) at least one edge among $e, e_1, e_2, \ldots e_k$ which is not contained in B (resp. T). Thus $B \setminus U'$ (resp. $T \setminus U'$) is also a d-blocker (resp. d-transversal) with cardinality at most k. Therefore, any d-blocker (resp. d-transversal) in G' with cardinality at most k can be transformed into a d-blocker (resp. d-transversal) in G with cardinality at most k and not using any edge of U. Conversely, any d-blocker (resp. d-transversal) in G not containing edges of U is also a d-blocker (resp. d-transversal) in G'.

However, the auxiliary graph G' has parallel edges. Since we want to show complexity results which even hold for simple graphs, we will describe another transformation of the graph G. Instead of introducing parallel edges for each edge $uv \in U$, we replace each edge of U by the following construction which we call a k-gadget (between u and v): we add a complete bipartite graph $K_{k+1,k+1} = (X, Y, W)$ and we link u to all vertices in X as well as v to all vertices in Y (see Fig. 4). The vertices u and v are called the endpoints of the k-gadget. We denote the graph obtained in this way by G''. The problem *BLOCK*(G, d, k) (resp. *TRANS*(G, d, k)) is equivalent to *BLOCK*(G'', d, k) (resp. *TRANS*(G'', d, k)) by the following observation. Let U'' be the set of edges contained in all k-gadgets used in G''. Notice that a k-gadget contains $k + 1$ disjoint perfect matchings and $k + 1$ disjoint matchings of cardinality $k + 1$ that do not saturate the endpoints. Therefore, for any d-blocker B (resp. d-transversal T) in G'' with $|B| \leq k$ (resp. $|T| \leq k$), every k-gadget contains a maximum matching using no edges of B (resp. T) as well as a matching with cardinality $k + 1$ not saturating the endpoints and not containing any edge in B (resp. T). Thus, $B \setminus U''$ (resp. $T \setminus U''$) is also a d-blocker (resp. d-transversal) with cardinality at most k. We conclude that any d-blocker (resp. d-transversal) in G'' with cardinality at most k can be transformed into a d-blocker (resp. d-transversal) in G with cardinality at most k and not using any edge of U. Conversely, any d-blocker (resp. d-transversal) in G not containing edges of U is also a d-blocker (resp. d-transversal) in G''.

The following proposition is an intermediate result used for proving the main complexity results stated afterwards.

Proposition 3.1. *Let $k \geq 4$ be an integer and let $G = (X, Y, E)$ be a simple bipartite graph such that*

1. $|X| > k$;
2. $|Y| = \binom{k}{2}$;
3. $d(y) = 2, \forall y \in Y$ *and* $d(x) \geq 1, \forall x \in X$;
4. *G contains no C_4.*

Then $\nu(G) \geq k + 1$.

Proof. This is equivalent to the following statement: In a simple graph $\widehat{G} = (X, E)$ without isolated vertices, with $|X| = q \geq k + 1$ and $k(k-1)/2$ edges, one can find a partial graph H where each connected component has at most one cycle and with $|E(H)| \geq k + 1$. Indeed starting from the vertex set X of G, we associate with every $y \in Y$ with neighbors $x'(y), x''(y)$ an edge $x'(y)x''(y)$. Since G contains neither C_4's nor multiple edges, the graph $\widehat{G}$ obtained in this way is a simple graph. Clearly there is a one-to-one correspondence between the matchings M in G and the partial graphs H of $\widehat{G}$ where each connected component has at most one cycle: for each edge x_iy_j of M in G, we orient the edge of $\widehat{G}$ associated to vertex y_j towards x_i. A matching M in G corresponds to a partial oriented graph $\widehat{H}$ in $\widehat{G}$ such that there is at most one arc entering into each vertex. Such an orientation exists if and only if every connected component of $\widehat{H}$ has at most one cycle. Let n_0 (resp. n_1) be the number of vertices of $\widehat{G}$ in connected components without cycles (resp. with cycles). Since each connected component on $\overline{n}$ vertices has $\overline{n} - 1$ (resp. $\overline{n}$) edges if it has no (resp. one) cycle, we only have to show that $\widehat{G}$ has at most $q - (k + 1)$ components which are trees. Indeed in such a $\widehat{G}$ we can then choose n_1 edges in connected components with cycles and $n_0 - q + k + 1$ edges in the connected components which are trees. This gives us a partial graph $\widehat{H}$ of $\widehat{G}$ with at least $n_1 + n_0 - (q - (k + 1)) = k + 1$ edges. Then $n_0 + n_1 = |X| = q$; if $n_1 \geq k + 1$ we are done: we can get a partial graph $\widehat{H}$ with $|E(\widehat{H})| \geq k + 1$. If $n_1 = k$, we are also done since $n_0 \geq 2$ and $\widehat{G}$ has no isolated vertex. So we can assume $n_1 < k$. Let by contradiction $\widehat{G}$ have more than $q - (k + 1)$ connected components which are trees. Then

$$|E(\widehat{G})| \leq \frac{n_1(n_1 - 1)}{2} + n_0 - (q - k) = \frac{n_1(n_1 - 1)}{2} + k - n_1$$
$$= \frac{n_1^2}{2} - \frac{3}{2}n_1 + k < \frac{k^2}{2} - \frac{3}{2}k + k = \frac{k(k-1)}{2},$$

a contradiction. $\square$

Theorem 3.2. *BLOCK(G, d, k) is $\mathcal{NP}$-complete when G is bipartite.*

Proof. The problem is clearly in $\mathcal{NP}$. To prove the $\mathcal{NP}$-completeness, we use a transformation from CLIQUE which is a well-known $\mathcal{NP}$-complete problem (see [9]). Let $G' = (V', E')$ be an undirected simple graph and let $r \leq |V'|$ be a positive integer. We construct a bipartite graph $G = (V, E)$ as follows: with each vertex $v'_i \in V'$, we associate a vertex $v_i \in V$ and with each edge $e'_{ij} = v'_iv'_j \in E'$ we associate a vertex $v_{ij} \in V$; for each vertex $v_{ij} \in V$ we add a new vertex $\bar{v}_{ij}$ as well as an edge $v_{ij}\bar{v}_{ij}$; finally for each edge $v'_iv'_j \in E'$, we add an edge v_iv_{ij} and an edge v_jv_{ij}.

Notice that the cardinality of a maximum matching M in G is $|M| = m$, where m is the number of edges in G'. Such a matching may be obtained by taking all the edges $v_{ij}\bar{v}_{ij}$. We will now prove the following statement which finishes the proof: G' contains a clique of size r if and only if there exists a $(\frac{r(r-3)}{2})$-blocker B in G with $|B| = \frac{r(r-1)}{2}$ and not using any edges of U, where $U = \cup_{v'_iv'_j \in E'} \{v_iv_{ij}, v_jv_{ij}\}$. Notice that the auxiliary graph obtained by replacing the edges of U by k-gadgets remains bipartite.

Let us suppose that G' contains a clique C of size r and let $E'_C \subseteq E'$ be the edges of this clique. By taking $B = \{v_{ij}\bar{v}_{ij} | e'_{ij} \in E'_C\}$, we obtain a $(\frac{r(r-3)}{2})$-blocker. In fact a maximum matching in the graph $G^* = (V, E \setminus B)$ is obtained by taking the remaining edges $v_{ij}\bar{v}_{ij}$ (there are exactly $m - \frac{r(r-1)}{2}$ such edges) and the edges of a maximum matching in the subgraph induced by vertices v_{ij} such that $e'_{ij} \in E'_C$ and the vertices v_i such that $v'_i \in C$ (the cardinality of such a matching is at most r). Thus $\nu(G^*) \leq m - \frac{r(r-1)}{2} + r = m - \frac{r(r-3)}{2}$.

Suppose now that there is a $(\frac{r(r-3)}{2})$-blocker B in G with $|B| = \frac{r(r-1)}{2}$ and not using any edges of U but there is no clique of size r in G'. This implies that the subgraph induced by vertices v_{ij} and the vertices v_i, v_j such that $v_{ij}\bar{v}_{ij} \in B$ is a simple bipartite graph $\tilde{G} = (X, Y, \tilde{E})$, where $Y = \{v_{ij} \in V \mid v_{ij}\bar{v}_{ij} \in B\}$ and X is the subset of the vertices $\{v_i \mid v'_i \in V'\}$ that are neighbors of Y in G. $\tilde{G}$ has the following properties:

(i) $|X| > r$ (because there is no clique of size r in G');
(ii) $|Y| = \frac{r(r-1)}{2} = \binom{r}{2}$;
(iii) $d(v_{ij}) = 2, \forall v_{ij} \in Y$ and $d(v_i) \geq 1, \forall v_i \in X$;

(iv) $\tilde{G}$ contains no C_4 (since there are no multiple edges in G');
(v) $\nu(\tilde{G}) \leq r$ (because all vertices in Y are saturated by any maximum matching in G and since B is a $(\frac{r(r-3)}{2})$-blocker, the cardinality of a maximum matching in $\tilde{G}$ is at most $|Y| - \frac{r(r-3)}{2} = r$).

Clearly (v) contradicts Proposition 3.1. Thus there must be a clique of size r in G' defined by the vertices v_i, v_j such that $v_{ij}\bar{v}_{ij} \in B$ and hence $|X| = r$. □

Remark 3.1. The proofs of Proposition 3.1 and Theorem 3.2 suggest to consider an alternative formulation. We may define for a graph G the value $\rho(G)$ which is the maximum number of edges in a unicyclic partial graph of G, where a graph is called unicyclic if every connected component has at most one cycle. By the above discussions we have for any graph G on $\binom{k}{2}$ edges that $\rho(G) \leq k$ if and only if G is a k-clique. It follows that determining whether for an arbitrary graph G, there is a subgraph H of G on $\binom{k}{2}$ edges with $\rho(H) \leq k$ is an $\mathcal{NP}$-complete problem since it is equivalent to deciding whether G contains a clique of size k.

Theorem 3.3. *BLOCK$(G, 1, k)$ is $\mathcal{NP}$-complete when G is bipartite.*

Proof. The claim will be proven by reducing $BLOCK(G', d, k)$ to $BLOCK(G, 1, k)$. Let $G' = (X', Y', E')$ be a bipartite graph, $d \in \{1, \ldots, \nu(G')\}$ and $k \in \{0, 1, \ldots, |E'|\}$. The graph $G = (X, Y, E)$ is defined as follows. $X = X'$ and $Y = Y' \cup Y'_a$ is the set Y' with $|X'| - \nu(G') + d - 1$ additional vertices $Y'_a = \{y'_1, \ldots, y'_{|X'|-\nu(G')+d-1}\}$. The set $E = E' \cup U$ consists of the edges in E' and for every pair of vertices $x \in X$ and $y' \in Y'_a$ we add an edge $[x, y]$. These added edges are denoted by U. Note that we have $\nu(G) = |X'|$ because of the following. Let M be a maximum matching in G'. In G the matching M can easily be completed to a matching with cardinality $|X'|$ by edges in U since $|Y'_a| \geq |X'| - \nu(G')$. In the following we prove that there is a d-blocker in G' with cardinality at most k if and only if there is a blocker in G with cardinality at most k and not using any edges of U. We begin by assuming that there is a d-blocker in G' with cardinality at most k and not using any edge of U. We will show that B is also a blocker for G. By contradiction assume that there is a matching M in $G \setminus B$ with cardinality $|X'|$. This implies that the set $M \setminus U$ is a matching in $G' \setminus B$ with cardinality at least $|X'| - |Y'_a| = \nu(G') - d + 1$ because any matching in G contains at most $|Y'_a|$ edges of U as U consists of $|Y'_a|$ bundles. This contradicts the fact that B is a d-blocker in G'.

Conversely suppose that there is no d-blocker in G' with cardinality at most k. Let $B \subseteq E'$ with $|B| \leq k$. Because there is no d-blocker in G' with cardinality at most k, there exists a matching $M \subseteq E' \setminus B$ with cardinality $\nu(G') - d + 1$. The matching M can be completed in G by edges in U to a matching M' with cardinality $|X'|$ since $|Y'_a| = |X'| - \nu(G') + d - 1$, implying that B is not a blocker in G. As B was chosen arbitrarily this implies that there does not exist a blocker in G with cardinality k and not using any edge of U. □

From Property 2.2 we deduce the following result.

Corollary 3.4. *TRANS$(G, d = 1, k)$ is $\mathcal{NP}$-complete when G is bipartite.*

Theorem 3.5. *For every fixed $d \in \{1, 2, \ldots\}$, TRANS(G, d, k) is $\mathcal{NP}$-complete when G is bipartite.*

Proof. First let us show that $TRANS(G, d, k)$ is in $\mathcal{NP}$. Given a set T of k edges, we assign a weight $w_1 = 1$ to these edges as well as a weight $w_2 = 1 + \frac{1}{m}$ to all the other edges in G, where m is the number of edges in G. If any maximum matching M in G has weight $W(M) \leq \nu(G)(1 + \frac{1}{m}) - \frac{d}{m}$, then T is necessarily a d-transversal since M uses d edges of T. Maximum matchings of maximum weight can be found in polynomial time, thus $TRANS(G, d, k)$ is in $\mathcal{NP}$.

We will reduce $TRANS(G', 1, k')$ to $TRANS(G, d, k)$. Let d be fixed in $\{1, 2, \ldots\}$, $G' = (X', Y', E')$ be a bipartite graph and $k' \in \{0, 1, \ldots, |E'|\}$. Let $G = (X, Y, E)$ be a bipartite graph defined as follows. X consists of the vertices in X' plus $d - 1$ additional vertices denoted by $\{x_1, \ldots, x_{d-1}\}$, Y consists of the vertices in Y' and $d - 1$ additional vertices denoted by $\{y_1, \ldots, y_{d-1}\}$ and $E = E' \cup \{x_i y_i \mid i \in \{1, \ldots, d - 1\}\}$. We will finally show that $TRANS(G', 1, k')$ is true exactly when $TRANS(G, d, k' + d - 1)$ is true. Suppose that T is a transversal in G' with $|T| = k'$. Then $T \cup \{x_i y_i | i \in \{1, \ldots, d - 1\}\}$ is a d-transversal in G with cardinality $k' + d - 1$, showing that $TRANS(G, d, k)$ evaluates to true with $k = k' + d - 1$. Conversely, suppose that T is a d-transversal in G with cardinality k. Without loss of generality we can assume that $E \setminus E' \subseteq T$ as $E \setminus E' \subseteq P_1(G)$. We therefore have that the set $T' = T \cap E'$ is a transversal in G' with cardinality $k' = k - d + 1$ showing that $TRANS(G', 1, k')$ evaluates to true. □

The proof of Theorem 3.5 can easily be adapted to the case of blockers, yielding the following theorem.

Theorem 3.6. *For every fixed $d \in \{1, 2, \ldots\}$, BLOCK(G, d, k) is $\mathcal{NP}$-complete when G is bipartite.*

4. Some special cases

We shall now examine some simple special cases of graphs for which minimum d-blockers and d-transversals can be found in polynomial time. Actually, we even give explicit formulae for the size of a minimum d-transversal and d-blocker.

The proofs of the following two results about chains and cycles are left to the reader.

Proposition 4.1. *Let $G = (V, E)$ be a chain on vertices $v_1, v_2, \ldots, v_n$ (i.e., $E = \{v_iv_{i+1} | i = 1, \ldots, n-1\}$) and $d \geq 1$ an integer. Then*

1. *$\beta_d(G) = 2d - 1$ and $\tau_d(G) = d$ if n is even,*
2. *$\beta_d(G) = \tau_d(G) = 2d$ if n is odd.*

Proposition 4.2. *Let $G = (V, E)$ be a cycle on vertices $v_1, v_2, \ldots, v_n$ (i.e., $E = \{v_iv_{i+1} | i = 1, \ldots, n-1\} \cup \{[v_n, v_1]\}$) and $d \geq 1$ an integer. Then*

1. *$\beta_d(G) = \tau_d(G) = 2d$ if n is even,*
2. *$\beta_d(G) = \tau_d(G) = 2d + 1$ if n is odd.*

Remark 4.1. Notice that in the case of an even chain, i.e., with an even number of edges, a minimum d-transversal is not necessarily composed of d bundles. In fact, a minimum 2-transversal in the chain $\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9\}$ may be obtained by taking edges $\{v_1v_2, v_3v_4, v_6v_7, v_8v_9\}$.

Now, we are interested in complete graphs and regular bipartite graphs.

Proposition 4.3. *Let $G = K_{n,n}$ be a complete bipartite graph. Then $T \subseteq E$ is a d-blocker if and only if T is a d-transversal.*

Proof. From Property 2.1, we only have to show that if T is a d-transversal, then it is also a d-blocker. So suppose that T is a d-transversal of G but it is not a d-blocker of G. This means that in the graph obtained from G by removing T one can find a maximum matching M of size at least $\nu(G) - d + 1 = n - d + 1$. Since G is a complete bipartite graph, M could easily be completed into a perfect matching M' in G by adding at most $d - 1$ edges from T. Thus $|M' \cap T| \leq d - 1$ which means that T is not a d-transversal which is a contradiction. □

Using similar arguments we obtain the following proposition.

Proposition 4.4. *Let $G = K_n$ be a complete graph. Then $T \subseteq E$ is a d-blocker if and only if T is a d-transversal.*

Proposition 4.5. *Let $G = (X, Y, E)$ be a Δ-regular bipartite graph. Then, if $|X| \geq d$, any set $B = \bigcup_{i=1}^{d} \omega(x_i) \subseteq E$, where $x_1, \ldots, x_d \in X$, is a minimum d-blocker.*

Proof. Clearly, after having removed B the cardinality of a maximum matching is at most $n - d$ since there are exactly d isolated vertices in X. Furthermore as G contains Δ disjoint maximum matchings, we have $\beta_d(G) \geq d\Delta$, as any d-blocker must contain at least d edges in each maximum matching. □

From Property 2.1, we deduce that $T = \bigcup_{i=1}^{d} \omega(x_i)$ is also a d-transversal for a Δ-regular bipartite graph $G = (X, Y, E)$. As a direct consequence of Property 2.3 we get the following result.

Proposition 4.6. *Let $G = (X, Y, E)$ be a Δ-regular bipartite graph. The minimum cardinality of a d-transversal T is $d\Delta$ (for any d with $1 \leq d \leq n$). Such a T may be constructed by taking:*

$$T = \omega(z_1) \cup \omega(z_2) \cup \cdots \cup \omega(z_d)$$

where $\{z_1, \ldots, z_d\} \subseteq X \cup Y$ is an independent set in G.

Proof. By Property 2.3, the set T as proposed is a d-transversal (with cardinality $d\Delta$). Furthermore as G contains Δ disjoint maximum matchings, the cardinality of any d-transversal is at least $d\Delta$. □

Theorem 4.7. *Let $d, n \geq 1$ be two integers with $2d \leq n$ and let $r = \lfloor \frac{n}{2} \rfloor - d$. For the graph K_n, let B be a d-blocker of minimum cardinality (B is also a minimum d-transversal of K_n by Proposition 4.4).*

1. *If $d \leq \lfloor \frac{n}{2} \rfloor - \frac{2}{5}n + \frac{3}{5}$, the cardinality of B is $\binom{n}{2} - \binom{2r+1}{2}$. B may be constructed by taking $n - 2r - 1$ bundles.*
2. *If $d \geq \lfloor \frac{n}{2} \rfloor - \frac{2}{5}n + \frac{3}{5}$, the cardinality of B is $\binom{n-r}{2}$. B may be constructed by taking a clique on $n - r$ vertices.*

Proof. Notice that searching for a minimum d-blocker B of K_n is equivalent to searching for a maximum partial graph $H = (V, E_H)$ of $K_n = (V, E)$ (i.e. a partial graph with n vertices and a maximum number of edges) such that $\nu(H) = r$. In fact, the edges not belonging to H will belong to B. Suppose that H is such a maximum partial graph corresponding to an $r < \lfloor \frac{n}{2} \rfloor$, i.e., $d \geq 1$. In the following we will prove various properties that H must satisfy to obtain eventually a complete description of the structure of H.

Claim: $\forall v \in S(H)$, v is connected to all other vertices of H.

Suppose by contradiction that $v \in S(H)$ and $u \in V$ with $vu \notin E_H$. Let H' be the graph obtained by adding vu to H. By edge-maximality of H we must have $\nu(H') = r + 1$. Let M' be a matching in H' with $|M'| = r + 1$. We have $vu \in M'$ as otherwise we would have $\nu(H) = r + 1$. Therefore $M' \setminus \{vu\}$ is a matching in H with cardinality r not saturating the vertex v. This violates $v \in S(H)$.

Claim: $\forall v \in V \setminus S(H)$, v is a simplicial vertex.

Let $v, u, w \in V$, with $v \notin S(H)$ and u, w are two distinct neighbors of v in H. Furthermore let M be a maximum matching in H which does not saturate v. Suppose by contradiction that $uw \notin E_H$. By edge-maximality of H this implies that the graph H' obtained by adding uw to H contains an augmenting chain with respect to the matching M. This augmenting chain consists of the edge uw and two alternating chains P_u and P_w in H where P_u has on the one end a non-saturated vertex and on the other end u and P_w has on the one end a non-saturated vertex and at the other end w. At most one of these two chains contains v. Suppose without loss of generality that the chain P_u does not go through v. This implies that if we append to the end u of the path P_u the edge uv, we obtain an augmenting chain in H contradicting the maximality of matching M.

The above claims imply that V can be partitioned into sets $C_0, C_1, \ldots, C_k$ with $C_0 = S(H)$, such that the subgraph of H induced by C_i is a clique $\forall i = 0, 1, \ldots, k$ and there is no edge in E_H connecting a vertex in C_i with a vertex in C_j for $1 \le i < j \le k$.

Claim: All sets $C_1, \ldots, C_k$ contain an odd number of vertices.

Suppose by contradiction that C_1 has an even number of vertices. Let $v \in C_1$. As $v \notin S(H)$, there exists a maximum matching M in H which does not saturate v. All other vertices in C_1 must be saturated as otherwise the matching M would not be maximum. As $C_1 \setminus \{v\}$ contains an odd number of vertices, at least one vertex $u \in C_1 \setminus \{v\}$ must be saturated by an edge $uw \in M$ with $w \in S(H)$. By replacing uw by uv in the matching M we get another maximum matching in H which does not saturate w. This contradicts $w \in S(H)$.

Claim: For any maximum matching M of H, no edge of M has both endpoints in $S(H)$.

Let M be a maximum matching in H and suppose by contradiction that there exists an edge $vu \in M$ with $v, u \in S(H)$. As $d \ge 1$, we have that at least two vertices in H are not saturated by M. Let $w \in V$ be such a non-saturated vertex. Replacing the edge vu by uw in the matching M, gives another maximum matching in H which does not saturate v and therefore contradicts $v \in S(H)$.

Claim: Let M be a maximum matching in H. For any set $C \in \{C_1, \ldots, C_k\}$ at most one edge in M goes from $S(H)$ to C.

Suppose by contradiction that there are two distinct edges $v_1u_1, v_2u_2 \in M$ with $v_1, v_2 \in S(H)$ and $u_1, u_2 \in C$. Let $w_1, w_2 \in V$ be two vertices which are not saturated by M. This implies that $(w_1v_1, v_1u_1, u_1u_2, u_2v_2, v_2w_2)$ is an augmenting chain in H with respect to M, thus contradicting the maximality of M.

Claim: The number of vertices in H not saturated by a maximum matching is $k - |S(H)|$.

By the above observations, a maximum matching M in H consists of at most $|S(H)|$ edges going from $S(H)$ to $C_1 \cup C_2 \cup \cdots \cup C_k$ and of edges linking two vertices of a same set C_i for $i \in \{1, 2, \ldots, k\}$. This implies that for every set $C \in \{C_1, \ldots, C_k\}$, M contains $\frac{|C|-1}{2}$ edges in C. Therefore every set C_i with $i \in \{1, 2, \ldots, k\}$ contains exactly one vertex which is not already saturated by the edges of M having both endpoints in C_i. Furthermore $|S(H)|$ of these k vertices will be saturated by edges of M incident to $S(H)$. The number of non-saturated vertices in H is therefore equal to $k - |S(H)|$.

Claim: There is at most one set in $\{C_1, \ldots, C_k\}$ containing more than one vertex.

Suppose by contradiction that $|C_1|, |C_2| > 1$. Let H' be the graph obtained from H by replacing the two sets C_1, C_2 by a set of size 1 and another set of size $|C_1| + |C_2| - 1$. By the previous claim, we have $\nu(H) = \nu(H')$. Furthermore H' contains more edges than H. This contradicts the edge-maximality of H.

Let C be the only set in $\{C_1, \ldots, C_k\}$ which may contain more than one vertex. By the above observations, the graph H can be characterized by two parameters, $p, q \in \{0, 1, \ldots\}$ where $p = |S(H)|$ and $2q + 1 = |C|$. The number of disjoint cliques in $H \setminus S(H)$ can be expressed by $k = n - p - 2q$. Therefore the number of vertices not saturated by a maximum matching in H is equal to $k - |S(H)| = n - 2p - 2q$ which must be equal to $n - 2r$ as the complement of graph H is a minimum d-blocker. This implies $r = p + q$ and allows us to describe H by one parameter p. We denote by $H(p) = (V_H(p), E_H(p))$ with $p \in \{0, \ldots, r\}$ this parametrized version of H. More precisely, $H(p)$ is the graph obtained by taking one clique on $2q + 1$ vertices where $q = r - p$, adding $n - p - (2q + 1)$ isolated vertices and finally adding a clique on p vertices (that corresponds to $S(H(p))$) which is connected to all other vertices. Note that by construction of $H(p)$ the cardinality of a maximum matching in $H(p)$ is independent of the parameter p. We are looking for the value of p, such that $H(p)$ has a maximum number of edges. We have $|E_H(p)| = p(n-1) - \binom{p}{2} + \binom{2q+1}{2} = \frac{3}{2}p^2 + (n - \frac{3}{2} - 4r)p + 2r^2 + r$ which is a strictly convex function of p. Therefore its maximum is attained at either $p = 0$ or $p = r$. Comparing these two cases we get that $|E_H(0)| \ge |E_H(r)|$ exactly when $d \le \lfloor \frac{n}{2} \rfloor - \frac{2}{5}n + \frac{3}{5}$ and $|E_H(0)| \le |E_H(r)|$ exactly when $d \ge \lfloor \frac{n}{2} \rfloor - \frac{2}{5}n + \frac{3}{5}$, thus finishing the proof. $\square$

Remark 4.2. The constructions suggested above are the only ones giving d-blockers of minimum cardinality, implying that only for the case $d = \lfloor \frac{n}{2} \rfloor - \frac{2}{5}n + \frac{3}{5}$, K_n contains two (and exactly two) non-isomorphic minimum d-blockers.

5. Conclusion

We have considered in this paper d-transversals and d-blockers in some special classes of graphs. The complexity of some basic problems related to blockers and transversals has been established. We have studied in particular the situation of (regular) bipartite graphs and of cliques. Additional cases where transversals and blockers can be found in polynomial time should be studied.

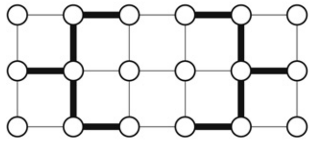

Fig. 5. The set $E \setminus T$ where T is a 7-transversal of $G_{3,6}$.

For instance the case of trees and of grid graphs should be examined. We recall that a grid graph $G_{m,n}$ has $m \times n$ vertices x_{ij} with integral coordinates ($1 \le i \le m$, $1 \le j \le n$) and (horizontal and vertical) edges linking vertices at distance 1.

It is interesting to observe that in grid graphs we may have minimum d-transversals which are not constructed by taking the bundles of vertices forming a stable set; moreover we may have for some values of d no minimum d-transversal consisting of bundles whose central vertices form a stable set. This is in particular the case in $G_{3,6}$ for minimum 7-transversals (see Fig. 5).

A maximum matching has $\frac{mn}{2} = 9$ edges. A set T of edges is a 7-transversal in $G_{3,6}$ if and only if no maximum matching has more than $9 - 7 = 2$ edges in $E \setminus T$ where E is the edge set of $G_{3,6}$. It is clearly the case for the set $E \setminus T$ shown in Fig. 5. Here we have $|E| = 2mn - (m + n) = 27$ and $|T| = 27 - 10 = 17$. No collection of 7 bundles built on a stable set can have less than 18 edges as can be verified.

We shall study the case of grid graphs and of trees in a forthcoming paper.

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