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Representations of Weakly Multiplicative Arithmetic Matroids are Unique

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Abstract. An arithmetic matroid is weakly multiplicative if the multiplicity of at least one of its bases is equal to the product of the multiplicities of its elements. We show that if such an arithmetic matroid can be represented by an integer matrix, then this matrix is uniquely determined. This implies that the integral cohomology ring of a centered toric arrangement whose arithmetic matroid is weakly multiplicative is determined by its poset of layers. This partially answers a question asked by Callegaro–Delucchi.

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1. Introduction

An arithmetic matroid $\mathcal{A}$ is a triple (E, rk, m), where (E, rk) is a matroid on the ground set E with rank function rk and $m\colon 2^E \to \mathbb{Z}_{\geq 1}$ is the so-called multiplicity function [3,7]. In the representable case, i.e., when the arithmetic matroid is determined by a list of integer vectors, this multiplicity function records data such as the absolute value of the determinant of a basis.

Arithmetic matroids were recently introduced by D'Adderio and Moci [7]. They capture many combinatorial and topological properties of toric arrangements [5,6,10,13] in a similar way as matroids carry information about the corresponding hyperplane arrangement [14,19]. The study of arithmetic matroids can be seen as a step towards the development of combinatorial frameworks to study the topology of very broad classes of spaces that are complements of normal crossing divisors in smooth projective varieties. See the introduction

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of Ref. [6] for more details on this line of research. Toric arrangements and arithmetic matroids play an important role in the theory of vector partition functions, which describe the number of integer points in polytopes [8,11]. They also appear naturally in the study of cell complexes and Ehrhart theory of zonotopes [1,13,18].

Let $X \in \mathbb{Z}^{d\times N}$ be a matrix. The arithmetic matroid represented by X is invariant under a left action of $\mathrm{GL}(d,\mathbb{Z})$ on X, under multiplication of some of the columns by -1, and under permutations of the columns. Therefore, when we are saying that a representation is unique, we mean that any two distinct representations are equal up to these three types of transformations.

An arithmetic matroid is *torsion-free* if $m(\emptyset) = 1$. Let $\mathcal{A} = (E, \mathrm{rk}, m)$ be a torsion-free arithmetic matroid. Let $B \subseteq E$ be a basis. We say that B is *multiplicative* if it satisfies $m(B) = \prod_{x\in B} m(\{x\})$. This condition is always satisfied if $m(B) = 1$. We call a torsion-free arithmetic matroid *weakly multiplicative* if it has at least one multiplicative basis. This notion was introduced in Ref. [12].

Theorem 1.1. *Let $\mathcal{A} = (E, \mathrm{rk}, m)$ be an arithmetic matroid of rank d that is weakly multiplicative, torsion-free, and representable. Then, $\mathcal{A}$ has a unique representation; i.e., if $X \in \mathbb{Z}^{d\times N}$ and $X' \in \mathbb{Z}^{d\times N}$ both represent $\mathcal{A}$, then there is a matrix $T \in \mathrm{GL}(d,\mathbb{Z})$, a diagonal matrix $D \in \mathbb{Z}^{N\times N}$ with diagonal entries in $\{1,-1\}$, and a permutation matrix $P \in \mathbb{Z}^{N\times N}$, such that $X' = TXDP$.*

Callegaro and Delucchi [6, Theorem 7.2.1] have recently put forward an incorrect proof[1] of this theorem in the special case where one basis has multiplicity 1.

Let $X \in \mathbb{Z}^{d\times N}$. Each column of X defines a character $\chi\colon (\mathbb{C}^*)^d \to \mathbb{C}^*$ of the complex torus $(\mathbb{C}^*)^d$. The set of kernels of these characters is called the centered toric arrangement defined by X. Callegaro and Delucchi [6] asked whether the isomorphism type of the integral cohomology ring of the complement of a complexified toric arrangement is determined combinatorially, i.e., by the poset of layers of the toric arrangement. Since the poset of layers encodes the arithmetic matroid [13, Lemma 5.4], Theorem 1.1 implies an affirmative answer in the special case of centered toric arrangements whose arithmetic matroid is weakly multiplicative.

The condition that the arithmetic matroid of the arrangement is weakly multiplicative can also be explained geometrically. Let us consider a centered toric arrangement $\mathcal{T}_X = \{\chi_1^{-1}(1), \ldots, \chi_N^{-1}(1)\}$, where each χ_i denotes a character. We assume that $\mathcal{T}_X$ is essential, i.e., $\bigcap_{i=1}^N \chi_i^{-1}(1)$ is 0-dimensional. Then, the arithmetic matroid corresponding to $\mathcal{T}_X$ is weakly multiplicative if and only if the following condition is satisfied: there is a set $I \subseteq [N]$ of cardinality

[1] In the proof in [6], the argumentation in Case b) is flawed. For example, the proof fails for the matrix $X = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{pmatrix}$. In the inductive step, it is claimed that the bottom right entry of X can be made positive, while all other signs are preserved (Case b)). This is false.

d, such that $\bigcap_{i\in I}\chi_i^{-1}(1)$ is 0-dimensional (i.e., I is a basis of the corresponding matroid) and the number of connected components of the intersection $\bigcap_{i\in I}\chi_i^{-1}(1)$ is equal to the product of the numbers of connected components of the $\chi_i^{-1}(1)$ for $i \in I$.

Corollary 1.2. *Let $\mathcal{T}_X$ be a centered toric arrangement in $(\mathbb{C}^*)^d$ whose corresponding arithmetic matroid is weakly multiplicative. Then, the integral cohomology ring of $\mathcal{T}_X$ is determined by its poset of layers.*

This result is a step towards a better understanding of one of the main problems in arrangement theory: to what extent is the topology of the complement of the arrangement determined by the combinatorial data?

Very recently, Pagaria pointed out that one can prove the following variation of Theorem 1.1, using essentially the same proof: if a representable arithmetic matroid $\mathcal{A} = (E, \mathrm{rk}, m)$ of rank d satisfies $m(E) = 1$, then it has a rationally unique representation, i.e., any two representations of $\mathcal{A}$ are equal up to left multiplication by a matrix $T \in \mathrm{GL}(d, \mathbb{Q})$, reversing the signs of the columns, and permuting the columns [16, Section 3].

On the other hand, the following example shows that, in our setting, there exist arithmetic matroids that are not weakly multiplicative and have several non-equivalent representations.

Example 1.3. For $a, b \in \mathbb{Z}$, we define the matrix:

$$X_{a,b} := \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}.$$

Let $b \geq 2$. Then, for any $a \in [b-1]$ that is relatively prime to b, the matrix $X_{a,b}$ is in Hermite normal form and it represents an arithmetic matroid $\mathcal{A}_b$ that is independent of a. $\mathcal{A}_b$ is the arithmetic matroid with underlying uniform matroid $U_{2,2}$, whose multiplicity function is equal to b on the whole ground set and 1 otherwise.

Remark 1.4. Theorem 1.1 does not only hold for "essential representations": suppose that $X \in \mathbb{Z}^{d\times N}$ and $X' \in \mathbb{Z}^{d\times N}$ both represent an arithmetic $\mathcal{A}$ of rank $r < d$ that is weakly multiplicative and torsion-free. We may assume that both matrices are in Hermite normal form. This implies that all non-zero entries are contained in the first r rows. Now, by Theorem 1.1, the submatrices consisting of the first r rows of X and X' are equal up to the usual transformations. Hence, X and X' are equal up to the usual transformations.

2. Background

2.1. Notation

We will use capital letters to denote matrices and the corresponding small letters to denote their entries. For $N \in \mathbb{N}$, we will write $[N]$ to denote the set $\{1, \ldots, N\}$. Usually, N will denote the cardinality of a set and d the dimension of the ambient space. We will always assume that $d \leq N$.

2.2. Arithmetic Matroids

We assume that the reader is familiar with the basic notions of matroid theory [15]. An arithmetic matroid is a triple (E, rk, m), where (E, rk) is a matroid and $m\colon 2^E \to \mathbb{Z}_{\geq 1}$ denotes the *multiplicity function*, which satisfies certain axioms. Since we are only discussing representable arithmetic matroids in this note, we do not list the axioms for the multiplicity function of an arithmetic matroid here. They can be found in Ref. [3].

A *representable arithmetic matroid* is an arithmetic matroid that can be represented by a finite list of elements of a finitely generated abelian group $G \cong \mathbb{Z}^d \oplus \mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_n}$. Representable and torsion-free arithmetic matroids can be represented by a finite list of elements of a lattice $G \cong \mathbb{Z}^d$. We will only consider this type of arithmetic matroid. We will assume that the ground set is always $E = \{e_1, \ldots, e_N\}$. Then, a list X of N vectors in $\mathbb{Z}^d$ can be identified with the matrix $X \in \mathbb{Z}^{d\times N}$ whose columns are the entries of the list.

A list of vectors $X = (x_e)_{e\in E} \subseteq \mathbb{Z}^d$ represents a vectorial matroid (E, rk) in the usual way, i.e., the rank function is the rank function from linear algebra. The multiplicity function m defined by X is defined as $m(S) := \left|(\langle S\rangle_{\mathbb{R}} \cap \mathbb{Z}^d)/\langle S\rangle\right|$ for $S \subseteq E$. Here, $\langle S\rangle \subseteq \mathbb{Z}^d$ denotes the subgroup generated by $\{x_e \colon e \in S\}$ and $\langle S\rangle_{\mathbb{R}} \subseteq \mathbb{R}^d$ denotes the subspace spanned by the same set. We will write $\mathcal{A}(X)$ to denote the arithmetic matroid that is represented by X.

Let $X \in \mathbb{Z}^{d\times N}$ and let $B \in \mathbb{Z}^{d\times d}$ be a submatrix of full rank. Slightly abusing notation, we will also write B to denote the corresponding basis of the underlying arithmetic matroid. It is well known (e.g. it is a special case of [18, Theorem 2.2]) that

$$m(B) = |\det(B)|\,. \tag{2.1}$$

It follows from the definition that, for $X \in \mathbb{Z}^{d\times N}$, $T \in \mathrm{GL}(d,\mathbb{Z})$, and $D \in \mathbb{Z}^{N\times N}$ a diagonal matrix whose diagonal entries are contained in $\{1, -1\}$, the matrices X and TXD represent the same arithmetic matroid. In other words, applying a unimodular transformation from the left and multiplying some columns by -1 does not change the arithmetic matroid that is represented by a matrix.

2.3. Hermite Normal Form

We say that a matrix $X \in \mathbb{Z}^{d\times N}$ of rank $r \leq d \leq N$ is in *Hermite normal form* if for all $i \in [d]$, $j \in [r]$, $0 \leq x_{ij} < x_{jj}$ for $i < j$ and $x_{ij} = 0$ for $i > j$, i.e., the first r columns of X form an upper triangular matrix and the diagonal elements are strictly bigger than the other elements in the same column. If $r < d$, this definition implies that the last $d - r$ rows of X are zero. It is not completely trivial, but well known, that any matrix $X \in \mathbb{Z}^{d\times N}$ of rank r can be brought into Hermite normal form by multiplying it from the left with a unimodular matrix $T \in \mathrm{GL}(d,\mathbb{Z})$ if the first r columns have rank r ([17, Theorem 4.1 and Corollary 4.3b]). Since such a multiplication does not change the arithmetic matroid represented by the matrix, we will be able to assume that a representation X of a torsion-free arithmetic matroid $\mathcal{A}$ is in Hermite normal form.

We recall the following simple lemma:

Lemma 2.1 ([12]). *Let $X \subseteq \mathbb{Z}^d$ be a list of vectors and let $B = (b_1, \ldots, b_d)$ be a multiplicative basis for the arithmetic matroid $\mathcal{A}(X) = (E, \mathrm{rk}, m)$. Let X' denote the Hermite normal form of X with respect to B. Then, the columns of X' that correspond to B form a diagonal matrix and the entries on the diagonal are $m(b_1), \ldots, m(b_d)$.*

2.4. Toric Arrangements

Let $T_{\mathbb{C}} := (\mathbb{C}^*)^d$ be the *complex* or *algebraic torus* and let $T_{\mathbb{R}} := (S^1)^d$ be the *real torus*. As usual, $S^1 := \{z \in \mathbb{C} \colon |z| = 1\}$. Each $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d$ determines a character of the torus, i.e., a map $\chi_\lambda \colon T_{\mathbb{C}} \to \mathbb{C}^*$ (or $T_{\mathbb{R}} \to S^1$ in the real case) via $\chi_\lambda(\phi_1, \ldots, \phi_d) := \phi_1^{\lambda_1} \cdots \phi_d^{\lambda_d}$. A *complex toric arrangement* in $T_{\mathbb{C}}$ is a finite set $\mathcal{T} = \{T_1, \ldots, T_N\}$ with $T_i := \chi_i^{-1}(a_i)$, where χ_i is a character and $a_i \in \mathbb{C}^*$ for all $i \in [N]$. A *real toric arrangement* is defined similarly: in this case, the χ_i are real characters and $a_i \in S^1$. A complex toric arrangement is called *complexified* if all a_i are contained in S^1. A toric arrangement is called *centered* if $a_i = 1$ holds for all $i \in [N]$. The set of characters defining a toric arrangement in the d-dimensional torus can be identified with a list of vectors in $\mathbb{Z}^d$. The arithmetic matroid represented by this list of vectors is the arithmetic matroid corresponding to the toric arrangement. A *layer* of a toric arrangement $\mathcal{T}$ is a connected component of a non-empty intersection of elements of $\mathcal{T}$. We obtain a poset structure on the set of layers of $\mathcal{T}$ by ordering them by reverse inclusion, i.e., $L \leq L'$ if $L' \subseteq L$.

3. Proof

We will prove Theorem 1.1 by carefully adapting and extending some methods that were developed by Brylawski and Lucas in an article on uniquely representable matroids. They showed that a representation X of a matroid over some field $\mathbb{K}$ is unique (up to certain natural transformations) if the entries of X are all contained in $\{0, 1, -1\}$ [4, Theorem 3.5].

Let $d \leq N$ be two integers. Let $\mathcal{A} = (E, \mathrm{rk}, m)$ be an arithmetic matroid that is represented by a matrix $X \in \mathbb{Z}^{d \times N}$. Without loss of generality, $E = [N]$. Furthermore, throughout this section, we assume that the ith column of X represents the element i of the matroid for all $i \in [N] = E$.

Let $B \subseteq E$ be a basis. We say that X is in B-*basic form* if there is a diagonal matrix $B \in \mathbb{Z}^{d \times d}$ of full rank with non-negative entries and $A \in \mathbb{Z}^{d \times (N-d)}$, such that $X = (B \,|\, A)$. Slightly abusing notation, we denote both the basis of $\mathcal{A}$ and the corresponding submatrix by B. By definition, as a basis of $\mathcal{A}$, $B = [d]$. We will index the columns of A by $d+1, \ldots, N$. If $\mathcal{A}$ is a weakly multiplicative arithmetic matroid that is represented by a matrix X, then we may assume by Lemma 2.1 that X is in B-basic form. In fact, X being in B-basic form is equivalent to X being in Hermite normal form and $[d]$ being a multiplicative basis.

Let C denote the matrix that is obtained from A by setting all non-zero entries to 1. This is called the *B-fundamental circuit incidence matrix*. This name is justified as follows: if we label the rows of C by $e_1, \ldots, e_d$ and the columns by $e_{d+1}, \ldots, e_N$, an entry c_{ij} of C is equal to 1 if and only if e_i is contained in the unique circuit contained in $B \cup \{e_j\}$, the so-called *fundamental circuit* of B and e_j.

The matrix C can also be seen as the adjacency matrix of a bipartite graph $\mathcal{G}_A$ with vertex set $\{r_1, \ldots, r_d\} \cup \{c_{d+1}, \ldots, c_N\}$, where r_i corresponds to the ith row and c_j corresponds to the jth column. It will be important that one can identify an edge $\{r_i, c_j\}$ of $\mathcal{G}_A$ with a non-zero entry a_{ij} of A. A spanning forest in this graph will be called a *coordinatizing path*. Let us fix a forest F in $\mathcal{G}_A$. Let c be an edge that is not contained in F. The fundamental circuit of F and c is called a *coordinatizing circuit* for c. Note that the graph $\mathcal{G}_A$ has N vertices. Let $\kappa(A)$ denote its number of connected components. It is easy to see that every coordinatizing path has cardinality $N - \kappa(A)$.

Example 3.1. Note that the matrix $X \in \mathbb{Z}^{3\times 7}$ is in B-basic form:

$$X = \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -4 & 0 & 3 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 3 & 0 & 1 & -1 & -1 \end{pmatrix} \end{array} \qquad C = \begin{array}{c c} & \begin{array}{cccc} e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} & \begin{pmatrix} \boxed{1} & 0 & 1 & 0 \\ \boxed{1} & \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & \boxed{1} & \boxed{1} \end{pmatrix} \end{array},$$

where C is the adjacency matrix of the graph $\mathcal{G}_A$ in Fig. 1. The entries of C that are highlighted define a coordinatizing path which corresponds to the spanning forest F of $\mathcal{G}_A$. There are only two edges in the graph $\mathcal{G}_A$ that are not contained in the spanning forest: a_{16} and a_{27}. They define the coordinatizing circuits $\{a_{25}, a_{35}, a_{27}, a_{37}\}$ and $\{a_{14}, a_{24}, a_{25}, a_{35}, a_{36}, a_{16}\}$. To simplify notation, we have described the edges of $\mathcal{G}_A$ by the corresponding entries of A.

Using the method described in the proof of Lemma 3.2, we can obtain a matrix X' from X where all the elements of the coordinatizing path are positive. We first pick a vertex in $\mathcal{G}_A$ that has degree 1 in the spanning tree, which we remove from the graph. Then, we iterate this process until we obtain a graph that has no edges. This leads to the following sequence of vertices: $r_1, c_4, r_2, c_5, c_6, c_7$. We obtain the matrix X' by multiplying by -1 (in that

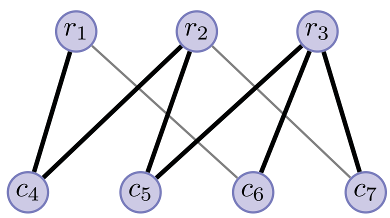

Figure 1. The bipartite graph corresponding to the matrix C in Example 3.1. The six edges contained in the spanning forest F are highlighted

order) column 7, column 6, row 1, and column 1. In three cases, the entry was already positive, so no rescaling was necessary:

$$X' = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 4 & 0 & 3 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 3 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

Lemma 3.2. *Let $X \in \mathbb{Z}^{d\times N}$ be a matrix of full rank d. Suppose that X is in B-basic form, i.e., there is a diagonal matrix $B \in \mathbb{Z}^{d\times d}$ of full rank with non-negative entries and $A \in \mathbb{Z}^{d\times(N-d)}$, such that $X = (B \mid A)$. Let $P = \{p_1, \ldots, p_{N-\kappa(A)}\}$ be a coordinatizing path and let $\sigma \in \{-1,1\}^{N-\kappa(A)}$. Then, there is a matrix $X' = (B \mid A')$ that represents the same arithmetic matroid $\mathcal{A}(X)$ and the entry p_j of A' is equal to σ_j times the corresponding entry of A. The matrix X' can be obtained from X by a sequence of multiplications of rows and columns by -1.*

Proof. This lemma is a modified version of Ref. [4, Proposition 2.7.3] and we are proving it in a similar way. The proof is by induction on $|P|$. If $|P| = 0$, there is nothing to prove. Let us assume that we have proved that the statement is true for all matrices $\tilde{A}$ that have a coordinatizing path $\tilde{P}$ with $|\tilde{P}| < k$. Suppose that $|P| = k \geq 1$. Since every forest that contains at least one edge has a vertex of degree one, there is some $a_{ij} = p_s \in P$ which is the unique entry common to P and some line (row r_i or column c_j) of A. Assume that line is row r_i. Then, deleting that row from A one easily sees that $\tilde{P} = P \setminus \{p_s\}$ is a coordinatizing path for the matrix obtained from A by deleting row r_i. By induction, we are able to change the signs of the entries of $\tilde{P}$ as prescribed by σ by multiplying rows and columns by -1 (which we may perform in A), giving $p_s = a_{ij}$ the value τa_{ij} for some $\tau \in \{-1, 1\}$. If we then multiply row r_i in A by $\sigma_s \tau$, we assign p_s the appropriate sign and we affect none of the entries of the coordinatizing path $\tilde{P}$ that were previously considered.

Since multiplying rows and columns of a matrix X by -1 does not change the arithmetic matroid $\mathcal{A}(X)$, both X and X' represent the same arithmetic matroid. $\square$

Lemma 3.3. *Let $X \in \mathbb{Z}^{d\times N}$ be a matrix of full rank d that represents an arithmetic matroid $\mathcal{A}$. Suppose that X is in B-basic form, in particular $X = (B \mid A)$. Suppose that $X' = (B \mid A') \in \mathbb{Z}^{d\times N}$ represents the same arithmetic matroid. Then, the entries of A and A' are equal up to sign, i.e., $|a_{ij}| = |a'_{ij}|$.*

Proof. Recall that the columns of A and A' are labeled by $d+1, \ldots, N$. For $j \in \{d+1, \ldots, N\}$, the set $\{e_1, \ldots, \hat{e}_i, \ldots, e_d, e_j\}$ is dependent if and only if the determinant of the corresponding submatrices of X and X' is 0. This holds if and only if $a_{ij} = a'_{ij} = 0$. If the set is independent, i.e., it is a basis, by (2.1):

$$|a_{ij}| = \frac{m\left(\left\{1, \ldots, \hat{i}, \ldots, d, j\right\}\right)}{\prod_{\nu\in[d]\setminus\{i\}} b_{\nu\nu}} = \left|a'_{ij}\right|.$$

$\square$

Lemma 3.4. *Let $X \in \mathbb{Z}^{d\times N}$ be a matrix of full rank d that represents an arithmetic matroid $\mathcal{A}$. Suppose that X is in B-basic form, in particular $X = (B \mid A)$. Then, up to sign, any non-zero subdeterminant of A is determined by the arithmetic matroid $\mathcal{A}(X)$.*

Proof. Let $I \subseteq [d]$ and $J \subseteq \{d+1, \ldots, N\}$ be two sets of the same cardinality. Let S be the submatrix of A whose rows are indexed by I and whose columns are indexed by J. If $\det(S) \neq 0$, then $B' := ([d] \setminus I) \cup J$ is a basis. It follows from (2.1) that $m(B') = |\det(S)| \prod_{\nu\in[d]\setminus I} b_{\nu\nu}$. Of course, $b_{\nu\nu}$ is equal to the multiplicity of the νth column of B. □

Lemma 3.5. *The matrix X' in Lemma 3.2 is uniquely determined.*

Proof. This proof uses some ideas of the proof of Ref. [4, Theorem 3.2]. Let $X'' = (B \mid A'')$ be another matrix that satisfies the consequence in Lemma 3.2. In particular, we assume that the entries of A' and A'' in the coordinatizing path are equal. By Lemma 3.3, the entries of A, A', and A'' must be equal up to sign. Hence, it is sufficient to show that all non-zero entries of A' and A'' that are not contained in the coordinatizing path are equal.

Recall that C denotes the B-fundamental circuit incidence matrix. Let us consider a non-zero entry α of C that is not contained in the coordinatizing path. α is contained in a unique coordinatizing circuit $\mathcal{C}$. Let a_1 and a_2 denote the entries of A' and A'' that correspond to α.

Suppose first that $|\mathcal{C}| = 4$. Then, $\mathcal{C}$ corresponds to a (2×2)-submatrix of A' or A'', respectively. The three other entries besides a_1 or a_2 are contained in the coordinatizing path P, and therefore, by assumption, they are equal for A' and A''. We will denote these three entries by b, c, and d. Then (up to relabelling the entries), the determinants of the two submatrices are $a_1d - bc$ and a_2d-bc, respectively. Since X' and X'' define the same arithmetic matroid, it follows from Lemma 3.4 that the absolute values of the two determinants must be equal. Now, suppose that $a_1 = -a_2$. Then, $|a_1d - bc| = |(-a_1)d - bc|$ must hold. This is equivalent to $a_1d - bc = -a_1d - bc$ or $a_1d - bc = a_1d + bc$. Both cases are impossible if all four number are non-zero. Hence, $a_1 = a_2$.

Let P_2 be the union of the coordinatizing path P with the coordinatizing circuit $\mathcal{C}$. We have determined all the entries in P_2 uniquely. Now, by an analogous argument, we can uniquely determine all entries of $C \setminus P_2$ which complete a circuit of size 4 in $\mathcal{G}_A$ with elements of P_2. Continuing this process, we end by uniquely determining all entries which can be attained by a sequence of circuits of size 4, three of whose members having been previously determined. We call the resulting set of determined entries P_2^*.

Now, let $\alpha \in C \backslash P_2^*$ be an entry that completes a circuit $\mathcal{C}$ of size 6 in $\mathcal{G}_A$ with elements of P_2^*. The circuit $\mathcal{C}$ corresponds to a 3×3 submatrix S of C. Again, let a_1 and a_2 denote the entries of A' and A'' that correspond to α. There are two cases to consider:

1. The 3×3 submatrix S has for its non-zero entries only the 6 entries of $\mathcal{C}$. In this case, S has two non-zero entries in each row and column. Hence, it is the sum of two permutation matrices. This implies that the

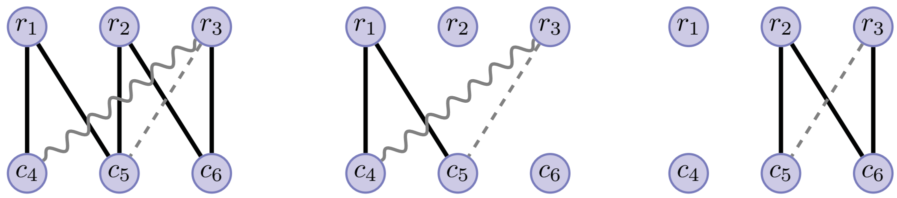

FIGURE 2. A coordinatizing circuit is short-circuited as in the proof of Lemma 3.5. The edges in P_2^* are shown in black. The wavy edge is α and the dashed edge is β

corresponding subdeterminants of A' and A'' are equal to $a_1x + y$ and a_2x+y, respectively, for some $x, y \neq 0$. As above, it is easy to see that it is not possible to have $a_1 = -a_2$ (which implies $|a_1x + y| = |(-a_1)x + y|$) if $a_1, x, y \neq 0$.

2. If there is another non-zero entry β in S, then β represents an additional edge which short-circuits the circuit $\mathcal{C}$ in the sense that it cuts across $\mathcal{C}$ to form a θ-graph.[2] Thus, β completes two smaller circuits with $\mathcal{C} \cup \{\beta\}$, one containing some previously determined elements and β, the other containing α and β. The former circuit implies that $\beta \in P_2^*$. Hence, the latter circuit shows that $\alpha \in P_2^*$, as well. See Fig. 2 for an example of this setting. □

We iterate the above argument to prove that the entries of A' and A'' that are contained in P_3^* must be equal, where P_3^* denotes the set of all non-zero entries of C which can be attained from P by a sequence of circuits of size $2t$ for $t \leq 3$. We define P_k^* analogously and assume that we have uniquely determined all entries of P_k^* for $k < m$. If $\alpha \in C \setminus P_{m-1}^*$ and α completes a circuit $\mathcal{C}$ of size $2m$ with entries from P_{m-1}^*, then there are two cases:

(1) The $m \times m$ submatrix S of C corresponding to the rows and columns of $\mathcal{C}$ has no non-zero entries other than those of $\mathcal{C}$. Then, as above, S is the sum of two permutation matrices and the corresponding subdeterminants of A' and A'' are equal to $a_1x + y$ and $a_2x + y$ for some $x, y \neq 0$, which implies that $a_1 = a_2$.

(2) If S contains another non-zero entry β, then $\mathcal{C} \cup \{\beta\}$ is a θ-subgraph of $\mathcal{G}_A$. Therefore, using the same argument as in the (3×3)-case, by induction, it follows that the entries of A' and A'' that correspond to α must be equal. □

Proof of Theorem 1.1. By assumption, the matrices X and X' both have full rank d. Let B be a basis that is weakly multiplicative. By Lemma 2.1, we may assume that both X and X' are in B-basic form, i.e., $X = (B \mid A)$ and $X' = (B \mid A')$ for suitable matrices A, A', and B. Lemma 3.3 implies that the entries of A and A' must be equal up to sign.

[2] Recall that a θ-graph is a graph that resembles the letter θ, i.e., it is the union of three internally disjoint paths that have the same two distinct end vertices.

Let P be a coordinatizing path. By Lemma 3.2, we may assume that the entries of A and A' that are contained in P are equal, after multiplying some rows and columns of X by -1. We are permitted to do these operations: recall that multiplying a row of X by -1 corresponds to multiplying X from the left with a certain matrix in $\mathrm{GL}(d, \mathbb{Z})$. Multiplying a column of X by -1 corresponds to multiplying X from the right with a certain non-singular diagonal matrix with diagonal entries in $\{1, -1\}$. We conclude by observing that Lemma 3.5 implies that all the remaining entries of A and A' must be equal too. □

4. Arithmetic Matroid Strata of the Integer Grassmannian

In this section, we will use the results in this paper to describe certain "strata" of an integer analog of the Grassmannian.

Recall that, for a matrix $A \in \mathbb{Z}^{d\times(N-d)}$, $\kappa(A)$ denotes the number of connected components of the bipartite graph with adjacency matrix A. If $X = (B|A) \in \mathbb{Z}^{d\times N}$ is in B-basic form, then this number is equal to the number of connected components of the matroid defined by X. A matroid is connected if for any two elements of the ground set, there is a circuit that contains both. Any matroid can be written as a direct sum of its connected components. This decomposition is unique up to isomorphism [15, Chapter 4].

We obtain the following result by combining Lemmas 3.2, 3.3, and 3.5.

Proposition 4.1. *Let $X = (B \mid A) \in \mathbb{Z}^{d\times N}$ with $B \in \mathbb{Z}^{d\times d}$ a diagonal matrix of full rank d and $A \in \mathbb{Z}^{d\times(N-d)}$. Let P be a coordinatizing path. For each of the $2^{N-\kappa(A)}$ possible choices of signs of the entries of P, there is a unique matrix $X_\sigma = (B \mid A_\sigma)$ with these signs that represents the same arithmetic matroid $\mathcal{A}(X)$.*

All representations of $\mathcal{A}(X)$ that are in B-basic form for this basis B can be obtained in this way.

Grassmannians are fundamental objects in algebraic geometry (e.g. [2, 9]). For a field $\mathbb{K}$, the Grassmannian $\mathrm{Gr}_{\mathbb{K}}(d, N)$ can be defined as the set of $(d \times N)$-matrices over $\mathbb{K}$ of full rank modulo a left action of $\mathrm{GL}(d, \mathbb{K})$. Similarly, one can define the integer Grassmannian $\mathrm{Gr}_{\mathbb{Z}}(d, N)$ as the set of all matrices $X \in \mathbb{Z}^{d\times N}$ of full rank, modulo a left action of $\mathrm{GL}(d, \mathbb{Z})$. The set of representations of a fixed torsion-free arithmetic matroid $\mathcal{A}$ of rank d on N elements is a subset of $\mathbb{Z}^{d\times N}$ that is invariant under a left action of $\mathrm{GL}(d, \mathbb{Z})$ and a right action of diagonal $(N \times N)$-matrices with entries in $\{\pm 1\}$, i.e., of $(\mathbb{Z}^*)^N$, the maximal multiplicative subgroup of $\mathbb{Z}^N$. This leads to a stratification of the integer Grassmannian $\mathrm{Gr}_{\mathbb{Z}}(d, N)$ into arithmetic matroid strata $\mathcal{R}(\mathcal{A}) = \{\bar{X} \in \mathrm{Gr}_{\mathbb{Z}}(d, N) \colon X \text{ represents } \mathcal{A}\}$. Here, we require that $\mathcal{A}$ has the ordered ground set $[N]$ and we consider ordered representations where the ith column of the matrix represents the element i. Proposition 4.1 allows us to calculate the cardinality of certain arithmetic matroid strata.

Corollary 4.2. *Let $\mathcal{A}$ be an arithmetic matroid of rank d on N elements that is weakly multiplicative and representable. Let $X = (B \mid A)$ be a representation in*

B-basic form. Then, the arithmetic matroid stratum of $\mathcal{A}$ of the integer Grassmannian $\mathrm{Gr}_{\mathbb{Z}}(d, N)$ has $2^{N-\kappa(\mathcal{A})}$ elements, where $\kappa(\mathcal{A})$ denotes the number of connected components of the underlying matroid of $\mathcal{A}$.

In other cases, determining the cardinality of the arithmetic matroid stratum has a number-theoretic flavor: for the arithmetic matroid $\mathcal{A}_b$ that was defined in Example 1.3, it is equal to $\varphi(b)$, where φ denotes Euler's totient function.

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