ON LOCAL ZETA-INTEGRALS FOR $\mathrm{GSp}(4)$ AND $\mathrm{GSp}(4) \times \mathrm{GL}(2)$

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ABSTRACT. We prove that Novodvorsky's definition of local L-factors for generic representations of $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ is compatible with the local Langlands correspondence when the $\mathrm{GL}(2)$ representation is non-supercuspidal. We also give an interpretation in terms of Langlands parameters of the "exceptional" poles of the $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ L-factor, and of the "subregular" poles of $\mathrm{GSp}(4)$ L-factors studied in recent work of Rösner and Weissauer; and deduce consequences for Gan–Gross–Prasad type branching laws, either for reducible generic representations, or for irreducible but non-generic representations.

1. INTRODUCTION

In this note, we study the local L-factors associated to irreducible smooth representations $\pi \times \sigma$ of the group $\mathrm{GSp}(4, F) \times \mathrm{GL}(2, F)$, where F is a nonarchimedean local field of characteric 0 (corrsponding to the natural 8-dimensional representation of the L-group). These L-factors can be defined in several possible ways. Firstly, one can use the local Langlands correspondence of [GT11]; secondly, one can use Shahidi's method. Thirdly, supposing π and σ to be generic, one can use a local zeta-integral of Rankin–Selberg type introduced by Novodvorsky [Nov79]. It is shown in [GT11] that the first two constructions agree, and we shall denote the resulting L-factor simply by $L(\pi \times \sigma, s)$. However, it is not obvious whether the L-factor $L^{\mathrm{Nov}}(\pi \times \sigma, s)$ defined via Novodvorsky's integral agrees with $L(\pi \times \sigma, s)$.

Conjecture α. *For any generic irreducible representations π of $\mathrm{GSp}_4(F)$ and σ of $\mathrm{GL}_2(F)$, we have $L(\pi \times \sigma, s) = L^{\mathrm{Nov}}(\pi \times \sigma, s)$.*

The Novodvorsky integral formula plays a key role in our recent work with Pilloni et al [LPSZ21] on the p-adic interpolation of L-values for cuspidal automorphic representations of GSp_4 and $\mathrm{GSp}_4 \times \mathrm{GL}_2$, which gives a further incentive to study Conjecture α. The conjecture is known to hold in a substantial range of cases by work of Soudry [Sou84], which we recall as Theorem 5.3 below, but many other cases still remain open.

1.1. Compatibility of L-factors.

Our first new result is the following:

Theorem A. *Conjecture α holds under the additional assumption that the $\mathrm{GL}(2, F)$-representation σ be non-supercuspidal.*

The case of σ an irreducible principal series was established in [LPSZ21, Theorem 8.9(i)], so it remains to consider the case when σ is a special representation. Twisting π appropriately, we can assume that $\sigma = \mathrm{St}$ is the Steinberg representation, and the proof in this case will be given as Theorem 7.3 below.

Since this paper was initially posted on the Mathematics ArXiv, a complementary result was proved by Yao Cheng [Che21], showing that Conjecture α also holds if σ is supercuspidal and π has trivial central character (so π factors through $\mathrm{PGSp}_4(F) \cong \mathrm{SO}_5(F)$). In particular, combining Cheng's result and Theorem A of the present paper proves Conjecture α, for any σ, if the central character of π is a square in the group of characters of $F^\times$; this is Theorem 1.3 of [Che21]. We are optimistic that combining the methods of this paper and [Che21] may lead to a complete proof of Conjecture α in the near future.

1.2. Exceptional poles for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$.

In the analysis of Novodvorsky's L-factor, an important role is played by a partition of the set of its poles into *regular* and *exceptional* poles (Definition 5.6). Let π and σ be as in Conjecture α. One sees easily that a necessary condition for $s_0 \in \mathbf{C}$ to be an exceptional pole of $L(\pi \times \sigma, s)$ is that $\chi_\pi \chi_\sigma |\cdot|^{2s_0} = 1$. We propose the following conjecture:

Conjecture β**.** *If $s_0 \in \mathbf{C}$ is such that $\chi_\pi\chi_\sigma|\cdot|^{2s_0} = 1$, then s_0 is an exceptional pole of $L^{\mathrm{Nov}}(\pi \times \sigma, s)$ if and only if it is a pole of the ratio*

$$\frac{L(\pi \times \sigma, s)L(\pi \times \sigma, s+1)}{L(\pi \times \sigma \times \mathrm{St}, s + \frac{1}{2})}.$$

Equivalently (by Lemma 7.1 below), s_0 is an exceptional pole if and only if the 8-dimensional Weil–Deligne representation $\phi_\pi \otimes \phi_\sigma$ has a 1-dimensional unramified direct summand whose L-factor has a pole at s_0.

Our second new result, whose proof is intertwined with that of Theorem A, is the following:

Theorem B. *Conjecture β holds under the additional ssumption that σ be non-supercuspidal.*

1.3. **Subregular poles for** $\mathrm{GSp}(4)$**.** In order to prove Theorems A and B, we shall use a relation between Novodvorsky's zeta-integral for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ and a zeta-integral for $\mathrm{GSp}(4)$ studied by Piatetski-Shapiro [PS97], depending on a choice of (split) Bessel model of π. Rösner and Weissauer [RW17, RW18] have computed the Piatetski-Shapiro L-factors for all generic π, and verified that they coincide with the Langlands L-factors (independently of the choice of Bessel model). In their computations, an important role is played by the notion of a *subregular* pole of the $\mathrm{GSp}(4)$ L-factor (see Definition 4.8 below). The proof of our main theorems also gives a conceptual interpretation of subregular poles, which may be of independent interest:

Theorem C. *Let π be a generic irreducible representation of $\mathrm{GSp}(4, F)$ with central character χ_π; and let $s_0 \in \mathbf{C}$. Then s_0 is a subregular pole of $L(\pi, s)$ (for some choice of split Bessel model) if and only if one of the following two possibilities occurs:*

1. *s_0 is a pole of the ratio $\dfrac{L(\pi, s)L(\pi, s+1)}{L(\pi \times \mathrm{St}, s + \frac{1}{2})}$; equivalently, the Langlands parameter of π has a 1-dimensional unramified direct summand whose L-factor has a pole at s_0. In this case, we necessarily have $\chi_\pi|\cdot|^{2s_0+1} \neq 1$.*
2. *$\chi_\pi|\cdot|^{2s_0+1} = 1$ and $s_0 + \frac{1}{2}$ is an exceptional pole of $L(\pi \times \mathrm{St}, s)$; equivalently, the Langlands parameter of π has a 2-dimensional, self-dual direct summand isomorphic to an unramified twist of the Steinberg parameter, whose L-factor has a pole at s_0.*

That is, a pole is subregular precisely when it arises from a direct summand of the Langlands parameter which is either 1-dimensional, or 2-dimensional and self-dual.

Remark 1.1. Theorem C is a fairly straightforward consequence of the results of [RW18]. We include it here partly because it motivates the formulation of Conjectures β and δ, and more importantly, because Theorem C plays a major role in the proof of Theorem A. More precisely, we shall prove directly that an analogue of Theorem C holds with the Langlands L-factor in the denominator replaced by the Novodvorsky L-factor, and deduce Theorem A when σ is the Steinberg by comparing this with Theorem C. ◇

1.4. **Distinction of representations.** Our next result is an interpretation of exceptional poles in terms of H-invariant periods, where $H = \{(h_1, h_2) \in \mathrm{GL}(2, F) \times \mathrm{GL}(2, F) : \det(h_1) = \det(h_2)\}$ (which is naturally a subgroup of $\mathrm{GSp}(4, F)$, see Section 2 below). It is not hard to show (see Corollary 5.8 below) that if s_0 is an exceptional pole of $L(\pi \times \sigma, s)$, then we have $\mathrm{Hom}_H\left(\pi \otimes (|\cdot|^{s_0} \boxtimes \sigma), \mathbf{C}\right) \neq 0$.

Conjecture δ. *The dimension of $\mathrm{Hom}_H\left(\pi \otimes (|\cdot|^{s_0} \boxtimes \sigma), \mathbf{C}\right)$ is 1 if s_0 is an exceptional pole of $L^{\mathrm{Nov}}(\pi \times \sigma, s)$, and 0 otherwise.*

Theorem D. *Conjecture δ is true if at least one of the following conditions holds:*

- *σ is non-supercuspidal,*
- *the central character of π is a square.*

Remark 1.2. The combination of Conjectures β and δ is closely related to the Gan–Gross–Prasad conjecture for non-tempered representations formulated in [GGP20].

More precisely, taking $s_0 = 0$, Conjectures β and δ predict that $\mathrm{Hom}_H\left(\pi \otimes (\mathbb{1} \boxtimes \sigma), \mathbf{C}\right)$ is non-zero if and only if the GSp_4-valued Weil–Deligne representation ϕ_π contains $\phi_\sigma^\vee$ as a self-dual direct summand. If we suppose $\chi_\pi = \chi_\sigma = 1$, so the representations involved factor through SO_5 and SO_4, then this condition on the Weil–Deligne representations is equivalent to the Langlands parameters of π and $\mathbb{1} \boxtimes \sigma^\vee$ forming a "relevant pair" in the sense of [GGP20]. According to the conjectures of *op.cit.*, this should be a necessary and sufficient condition for $\mathrm{Hom}_H\left(\pi \otimes (\mathbb{1} \boxtimes \sigma), \mathbf{C}\right)$ to be non-zero.[1]

So, in the light of Theorem D, Conjecture β is an instance of the non-tempered Gan–Gross–Prasad conjectures (mildly generalised from orthogonal groups to spin groups); and Theorem B verifies the conjecture for representations of this type when σ is non-supercuspidal. $\diamond$

1.5. **Multiplicity one for reducible representations.** We now give an interpretation of the above results in terms of branching laws for reducible representations. It follows from results of Prasad and Emory–Takeda[2] that we have $\dim \mathrm{Hom}_H(\pi \otimes (\sigma_1 \boxtimes \sigma_2), \mathbf{C}) \leqslant 1$ for any irreducible generic representations π of $\mathrm{GSp}(4, F)$ and σ_1, σ_2 of $\mathrm{GL}(2, F)$. Of course, this Hom-space can only be non-zero if $\chi_\pi \chi_{\sigma_1} \chi_{\sigma_2} = 1$.

We consider here the situation in which one or both of the σ_i is replaced by the reducible principal-series representation Σ having the Steinberg representation as subrepresentation. (However, we continue to assume that π itself is irreducible and generic.) One checks easily that for any irreducible generic σ with $\chi_\pi \chi_\sigma = 1$, the leading term at $s = 0$ of the zeta-integral defining $L^{\mathrm{Nov}}(\pi \times \sigma, s)$ gives a non-zero element of $\mathrm{Hom}_H(\pi \otimes (\Sigma \boxtimes \sigma), \mathbf{C})$. Similarly, if $\chi_\pi = 1$, then the leading term of Piatetski-Shapiro's zeta integral (with $\lambda_1 = \lambda_2 = 1$ in the notation of Section 4.1) defines a nonzero element of $\mathrm{Hom}_H(\pi \otimes (\Sigma \boxtimes \Sigma), \mathbf{C})$. We conjecture that these Hom-spaces are actually 1-dimensional, giving a generalisation to $\mathrm{GSp}_4 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ of the results on branching laws for reducible representations proved in [HS01] and [Loe21]:

Conjecture ε.

- (a) *Suppose π and σ are irreducible and generic, with $\chi_\pi \chi_\sigma = 1$. Then $\mathrm{Hom}_H(\pi \otimes (\Sigma \boxtimes \sigma), \mathbf{C})$ is 1-dimensional (and hence the leading term of the Novodvorsky zeta-integral is a basis of this space).*
- (b) *Suppose π is irreducible and generic with $\chi_\pi = 1$. Then the space $\mathrm{Hom}_H(\pi \otimes (\Sigma \boxtimes \Sigma), \mathbf{C})$ is 1-dimensional (and hence the leading term of the Piatetski-Shapiro zeta-integral is a basis).*

We shall see in §9 below that Conjecture ε(a) implies Conjecture δ, and we shall prove the following partial result:

Theorem E.

- (a) *Conjecture ε(a) is true if at least one of the following two conditions holds:*
  - *(i) χ_π is a square in the group of characters of $F^\times$;*
  - *(ii) σ is non-supercuspidal, and $s = 0$ is not an exceptional pole of $L^{\mathrm{Nov}}(\pi \times \sigma, s)$.*
- (b) *Conjecture ε(b) is true.*

These results are used in [LZ20] and [LZ21] to study Euler systems for Shimura varieties attached to $\mathrm{GSp}(4)$ and $\mathrm{GSp}(4) \times \mathrm{GL}(2)$.

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2. General notation

We shall consider the following setting:

- F is a nonarchimedean local field of characteristic 0, and q is the cardinality of its residue field.
- $|\cdot|$ the absolute value on F, normalised by $|\varpi| = \frac{1}{q}$ for ϖ a uniformizer.

[1] In *op.cit.* it is also assumed that the L-parameters are "of Arthur type", which in this situation corresponds to assuming that π and σ are tempered; but this is not essential to the formulation of the conjecture. It suffices that π and σ are generic (or members of generic L-packets).

[2] The restriction $(\sigma_1 \boxtimes \sigma_2)|_H$ is a direct sum of irreducible H-representations lying in the same L-packet. Theorem 5 of [Pra96] shows that there is at most one representation τ in this L-packet such that $\mathrm{Hom}_H(\pi \otimes \tau, \mathbf{C}) \neq 0$; and the general result on multiplicity-one for GSpin groups from [ET23], via the isomorphisms $\mathrm{GSp}_4 \cong \mathrm{GSpin}_5$ and $H \cong \mathrm{GSpin}_4$, shows that for any such τ the Hom-space has dimension $\leqslant 1$, giving the claim. Alternatively, the multiplicity-one result can be extracted directly from the proof of [Pra96, Theorem 5] (Prasad, pers.comm.), although the result is not explicitly stated there.

- We fix a nontrivial additive character $e : F \to \mathbf{C}^\times$.
- G denotes the group $\mathrm{GSp}(4, F)$ of matrices preserving the standard anti-diagonal symplectic form, and H the group $\{(h_1, h_2) \in \mathrm{GL}(2, F) \times \mathrm{GL}(2, F) : \det(h_1) = \det(h_2)\}$. We consider H as a subgroup of G via the embedding
$$\iota : \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \mapsto \begin{pmatrix} a &&& b \\ & a' & b' & \\ & c' & d' & \\ c &&& d \end{pmatrix}.$$
- In this paper "representation" will mean an admissible smooth representation on a complex vector space.
- An "L-factor" will mean a function of $s \in \mathbf{C}$ of the form $1/P(q^{-s})$, where P is a polynomial with $P(0) = 1$. Any fractional ideal of $\mathbf{C}[q^s, q^{-s}]$ containing the unit ideal is generated by a unique L-factor.

3. Principal series representations of GL(2)

3.1. Definitions.

Definition 3.1. *For μ, ν smooth characters $F^\times \to \mathbf{C}^\times$, and $s \in \mathbf{C}$, we write $i_s(\mu, \nu)$ for the space of smooth functions $f : \mathrm{GL}(2, F) \to \mathbf{C}$ satisfying*
$$f\left(\left(\begin{smallmatrix} a & \star \\ 0 & d \end{smallmatrix} \right) g \right) = \mu(a)\nu(d)|a/d|^s f(g),$$
with $\mathrm{GL}(2, F)$ acting via right translation. If $s = \frac{1}{2}$ we write simply $i(\mu, \nu)$.

As is well known, $i(\mu, \nu)$ is irreducible unless $\mu/\nu = |\cdot|^{\pm 1}$; if $\mu/\nu = |\cdot|$ it has a 1-dimensional quotient, and if $\mu/\nu = |\cdot|^{-1}$ it has a 1-dimensional subrepresentation. There is a unique (up to scalars) non-zero intertwining operator $i_s(\mu, \nu) \to i_{1-s}(\nu, \mu)$. The *Steinberg representation* St is the unique irreducible subrepresentation of $i(|\cdot|^{1/2}, |\cdot|^{-1/2})$.

3.2. Godement–Siegel sections.

Let $\mathcal{S}(F^2)$ denote the Schwartz space of locally-constant, compactly-supported functions on F^2, with $\mathrm{GL}(2, F)$ acting via the usual formula $(g \cdot \Phi)(x, y) = \Phi((x, y) \cdot g)$. Then we define
$$f^{\Phi}(g; \mu, \nu, s) = \mu(\det g) |\det g|^s \int_{F^\times} \Phi((0, x) \cdot g)(\mu/\nu)(x)|x|^{2s} \, \mathrm{d}^\times x,$$
which converges for $\Re(s) > 0$ and defines an element of $i_s(\mu, \nu)$. We write simply $f^{\Phi}(\mu, \nu, s)$ for the function $f^{\Phi}(-; \mu, \nu, s)$. We may extend the definition to all $s \in \mathbf{C}$ by analytic continuation, away from simple poles at the s such that $|\cdot|^{2s} = \nu/\mu$.

Remark 3.2. We have $f^{\Phi}(g; \mu, \nu, s) = \mu(\det g) f^{\Phi}(g; \mu/\nu, s)$ in the notation of [LPSZ21, §8.1]. $\diamond$

Proposition 3.3. *Let $\widehat{\Phi}$ denote the Fourier transform.*

(i) If $\nu \neq 1$, then the map $\Phi \mapsto f^{\Phi}(1, \nu, 0)$ is well-defined, nonzero, and $\mathrm{GL}(2, F)$-equivariant, and identifies $i(|\cdot|^{-1/2}, |\cdot|^{1/2}\nu)$ with the maximal quotient of $\mathcal{S}(F^2)$ on which $F^\times$ acts by ν.

(ii) If $\nu \neq |\cdot|^{-2}$, then the map $\Phi \mapsto f^{\widehat{\Phi}}(\nu, 1, 1)$ is is well-defined, nonzero, and $\mathrm{GL}(2, F)$-equivariant, and identifies $i(|\cdot|^{1/2}\nu, |\cdot|^{-1/2})$ with the maximal quotient of $\mathcal{S}(F^2)$ on which $F^\times$ acts by ν.

Proof. Well-known. $\square$

3.3. Whittaker functions.

For $\Phi \in \mathcal{S}(F^2)$ and μ, ν smooth characters, we define
$$W^{\Phi}(g; \mu, \nu, s) = \int_F f^{\Phi}\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right) g; \mu, \nu, s \right) e(x) \, \mathrm{d}x,$$
and $W^{\Phi}(g; \mu, \nu) = W^{\Phi}(g; \mu, \nu, \frac{1}{2})$. Again we write simply $W^{\Phi}(\mu, \nu)$ for the function $W^{\Phi}(-; \mu, \nu)$. Note that the integral is entire as a function of s, although $f^{\Phi}(-)$ may not be, and there is no s such that $W^{\Phi}(g; \mu, \nu, s)$ vanishes for all g and Φ. We have
$$W^{\Phi}(\mu, \nu, s) = \varepsilon \cdot W^{\widehat{\Phi}}(\nu, \mu, 1 - s)$$
where ε is a nonzero constant independent of Φ (a local root number). We want to study the space of functions $W^{\Phi}(\mu, \nu)$ for varying $\Phi \in \mathcal{S}(F^2)$.

- If $\sigma = i(\mu, \nu)$ is irreducible, then the space of functions $W^{\Phi}(\mu, \nu)$ for varying $\Phi \in \mathcal{S}(F^2)$ is precisely the Whittaker model[3] $\mathcal{W}(\sigma)$ of σ.
- If σ has a one-dimensional quotient, then the functions $f^{\Phi}(\mu, \nu, s)$ are regular at $s = \frac{1}{2}$ and span the representation σ; and mapping f^{Φ} to W^{Φ} gives a bijection from σ to a subspace $\mathcal{W}(\sigma) \subset \operatorname{Ind}_{N_2}^{\mathrm{GL}_2} e^{-1}$, containing the Whittaker model of the generic subrepresentation σ^{gen} as a codimension-1 subspace.
- If σ has a one-dimensional subrepresentation, then it does not have a Whittaker model; and the functions $W^{\Phi}(\mu, \nu)$ instead give the Whittaker model of $\sigma' = i(\nu, \mu)$, as we have just defined it. In this case, the $f^{\Phi}(\mu, \nu, s)$ are not all well-defined at $s = \frac{1}{2}$ (they may have poles). If we define
$$\mathcal{S}_0(F^2) := \{\Phi \in \mathcal{S}(F^2) : \Phi(0,0) = 0\},$$
then the f^{Φ} for $\Phi \in \mathcal{S}_0(F^2)$ are well-defined and span σ. The corresponding W^{Φ} span the Whittaker model of the irreducible subrepresentation of σ', which is also the irreducible quotient of σ.

4. Bessel models

Throughout this section, π denotes an irreducible representation of G with central character χ_π.

4.1. **The Bessel model.** Let $\Lambda = (\lambda_1, \lambda_2)$ be a pair of characters of $F^\times$ with $\lambda_1 \lambda_2 = \chi_\pi$. A (split) *Bessel model* of π (with respect to Λ) is a G-invariant subspace isomorphic to π inside the space of functions $G \to \mathbf{C}$ satisfying
$$B\left(\begin{pmatrix} 1 & & u & v \\ & 1 & w & u \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} x & & & \\ & y & & \\ & & x & \\ & & & y \end{pmatrix} g\right) = e(u)\lambda_1(x)\lambda_2(y)B(g).$$
It follows from [RS16, Theorem 6.3.2(i)] that if such a subspace exists, it is unique, and we denote it by $\mathcal{B}_\Lambda(\pi)$.

4.2. **Piatetski-Shapiro's integral.** Suppose π admits a Λ-Bessel model $\mathcal{B}_\Lambda(\pi)$.

Definition 4.1. *For $B \in \mathcal{B}_\Lambda(\pi)$, μ a smooth character of $F^\times$, and $\Phi_1, \Phi_2 \in \mathcal{S}(F^2)$, we define*
$$Z(B, \Phi_1, \Phi_2; \Lambda, \mu, s) = \int_{N_H \backslash H} B(h) \Phi_1((0,1) \cdot h_1) \Phi_2((0,1) \cdot h_2) \mu(\det h) |\det h|^{s+1/2} \, \mathrm{d}h,$$
where $N_H = \left(\left(\begin{smallmatrix} 1 & \star \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \star \\ 0 & 1 \end{smallmatrix}\right)\right)$ is the unipotent radical of the standard Borel subgroup of H.

This converges for $\Re(s) \gg 0$ and has meromorphic continuation as a rational function of q^s. If μ is the trivial character, we write simply $Z(B, \Phi_1, \Phi_2; \Lambda, s)$; we can always reduce to this case by replacing π with $\pi \otimes \mu$, and (λ_1, λ_2) with $(\lambda_1 \mu, \lambda_2 \mu)$. The following is the main result of [RW17]:

Theorem 4.2 (Rösner–Weissauer)**.** *The $\mathbf{C}$-vector space spanned by the functions*
$$\{Z(B, \Phi_1, \Phi_2; \Lambda, s) : B \in \mathcal{B}_\Lambda(\pi), \Phi_1, \Phi_2 \in \mathcal{S}(F^2)\}$$
is a fractional ideal of $\mathbf{C}[q^s, q^{-s}]$ containing the constant functions. This ideal is independent of Λ, and is generated by the L-factor $L(\pi, s)$ associated to the Langlands parameter ϕ_π.

4.3. **Generic representations.** Recall that π is said to be *generic* if it admits a Whittaker model, i.e. if it is isomorphic to a G-invariant subspace of the space of functions $W : G \to \mathbf{C}$ satisfying
$$W\left(\begin{pmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x \\ & & & 1 \end{pmatrix} g\right) = e(x+y)W(g). \tag{1}$$
Such a model is unique if it exists; we denote it by $\mathcal{W}(\pi)$.

[3] We define all Whittaker models for $\mathrm{GL}(2, F)$ with respect to the character $\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) \mapsto e(-x)$; this is slightly non-standard, but will simplify our formulae later

Proposition 4.3. *Suppose π is generic, and let μ be a smooth character of $F^\times$. For any $W \in \mathcal{W}(\pi)$, the integral*

$$B(W;\mu,s) := \int_{F^\times}\int_F W\left(\begin{pmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1\end{pmatrix} w_2\right)|a|^{s-3/2}\mu(a)\,\mathrm{d}x\,\mathrm{d}^\times a, \qquad w_2 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1\end{pmatrix}$$

converges for $\Re(s) \gg 0$ and has meromorphic continuation as a rational function of q^s. The set $\{B(W;\mu,s) : W \in \mathcal{W}(\pi)\}$ is a fractional ideal of $\mathbf{C}[q^s,q^{-s}]$ containing the constant functions, and it is generated by the spinor L-factor $L(\pi \times \mu, s)$ associated to the Langlands parameter of $\pi \times \mu$.

Proof. The definition of the integral, and the proof of its analytic continuation, are due to Novodvorsky [Nov79]. The proof that the L-factor defined by this integral coincides with the Langlands L-factor is due to Takloo-Bighash [TB00]. □

Proposition 4.4 (Roberts–Schmidt)**.** *For any s, the space of functions*

$$\widetilde{B}_W(g;\mu,s) := \frac{1}{L(\pi\times\mu,s)}B(gW;\mu,s)$$

for $W \in \mathcal{W}(\pi)$ is the Bessel model $\mathcal{B}_\Lambda(\pi)$, where $\Lambda = (\mu^{-1}|\cdot|^{1/2-s}, \mu\chi_\pi|\cdot|^{s-1/2})$.

See [RS16] for details. Since μ is arbitrary, we see that a generic representation has a Bessel model for every character Λ with $\lambda_1\lambda_2 = \chi_\pi$.

4.4. **Exceptional and subregular poles.** Suppose π admits a Λ-Bessel model.

Definition 4.5. *We define $L^\Lambda_{\mathrm{reg}}(\pi,s)$ and $L^\Lambda_{\mathrm{Kir}}(\pi,s)$ as the unique L-factors such that*

$$\left(\left\{Z(B,\Phi_1,\Phi_2;\Lambda,s) : B \in \mathcal{B}_\Lambda(\pi), \Phi_1,\Phi_2 \in \mathcal{S}(F^2), \Phi_1(0,0)\Phi_2(0,0) = 0\right\}\right) = \left(L^\Lambda_{\mathrm{reg}}(\pi,s)\right)$$
$$\left(\left\{Z(B,\Phi_1,\Phi_2;\Lambda,s) : B \in \mathcal{B}_\Lambda(\pi), \Phi_1,\Phi_2 \in \mathcal{S}(F^2), \Phi_1(0,0) = \Phi_2(0,0) = 0\right\}\right) = \left(L^\Lambda_{\mathrm{Kir}}(\pi,s)\right)$$

We let $L^\Lambda_{\mathrm{ex}}(\pi,s) = L(\pi,s)/L^\Lambda_{\mathrm{reg}}(\pi,s)$, and $L^\Lambda_{\mathrm{sub}}(\pi,s) = L^\Lambda_{\mathrm{reg}}(\pi,s)/L^\Lambda_{\mathrm{Kir}}(\pi,s)$, which are clearly also L-factors, so we have

$$L(\pi,s) = L^\Lambda_{\mathrm{ex}}(\pi,s)\cdot L^\Lambda_{\mathrm{sub}}(\pi,s)\cdot L^\Lambda_{\mathrm{Kir}}(\pi,s).$$

The poles of $L^\Lambda_{\mathrm{ex}}(\pi,s)$ are said to be exceptional poles *for π and Λ; the poles of L^Λ_{sub} are said to be* subregular poles.

Remark 4.6. The factor we call $L^\Lambda_{\mathrm{Kir}}(\pi,s)$ is denoted by $L(s,M)$ in the works of Rösner–Weissauer, where M is a certain auxiliary space. The notation $L^\Lambda_{\mathrm{Kir}}(\pi,s)$ is intended to emphasise the relation with Kirillov models. ◇

Theorem 4.7 (Piatetski-Shapiro, [PS97, Theorem 4.3])**.** *If π is generic, then $L^\Lambda_{\mathrm{ex}}(\pi,s)$ is identically 1, for all possible choices of Λ.*

So exceptional poles do not occur for generic representations; however, we shall see later that subregular poles do frequently occur. The poles of $L^\Lambda_{\mathrm{sub}}(\pi,s)$ (if any) are simple [RW18, Corollary 3.2]. We say $s = s_0$ is a *type I subregular pole* if it is a pole of the ratio

$$\frac{Z(B,\Phi_1,\Phi_2;\Lambda,s)}{L^\Lambda_{\mathrm{Kir}}(\pi,s)}$$

for some (Φ_1,Φ_2) with $\Phi_1(0,0) = 0$, and a *type II subregular pole* if we may take (Φ_1,Φ_2) such that $\Phi_2(0,0) = 0$. Clearly, any subregular pole must be of type I or type II (but these possibilities are not mutually exclusive).

Since the two factors of H are conjugate in G, one checks that s_0 is a type II subregular pole for the (λ_1,λ_2) Bessel model if and only if it is a type I subregular pole for the (λ_2,λ_1) Bessel model. So it suffices to analyse type II subregular poles. Moreover, if s_0 is a type II subregular pole, then it must also be a pole of $L(\lambda_1, s+\frac{1}{2})$ (cf. Proposition 3.1 of [RW18]; note that the characters ρ and ρ^* of *op.cit.* are λ_2 and λ_1 in our notation – the order is switched since we use a different matrix model of GSp_4). In particular, for a given π whose L-factor has a pole at s_0, there is at most one character Λ such that s_0 is a type II subregular pole for the Λ-Bessel model, namely $\Lambda = \left(|\cdot|^{-1/2-s_0}, \chi_\pi|\cdot|^{1/2+s_0}\right)$.

Definition 4.8. *Suppose π is generic. We shall simply say "s_0 is a subregular pole of $L(\pi, s)$" to mean that it is a type II subregular pole for this specific Bessel character, or (equivalently) a type I subregular pole for the character given by swapping λ_1 and λ_2.*

Note that these two Bessel characters coincide if and only if $\chi_\pi |\cdot|^{2s_0+1} = 1$.

The subregular poles have been tabulated for all Bessel models in [RW17, RW18]. Non-supercuspidal representations of $\mathrm{GSp}(4, F)$ have been classified by Sally and Tadić [ST93], into 11 types I–XI; the tables in [RS07, Appendix A] are a useful reference. All types except I, VII, and X have several subtypes, with subtypes "a" being the generic representations. So the generic non-supercuspidal representations are those of Sally–Tadic types {I, IIa, IIIa, IVa, Va, VIa, VII, VIIIa, IXa, X, XIa}. We can neglect the supercuspidal representations and those of types {VII, VIIIa, IXa}, since $L(\pi, s)$ is identically 1 for all such representations.

Theorem 4.9 (Rösner–Weissauer)**.** *If π is a generic representation, then every pole of $L(\pi, s)$ is subregular, unless π is of type IIIa or IVa, in which case there are no subregular poles.* □

5. Zeta integrals for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$

5.1. **Novodvorsky's integral.** We now suppose π is a generic irreducible representation of G; and we let σ be a representation of $\mathrm{GL}_2(F)$ which is either irreducible and generic, or a reducible principal-series representation with one-dimensional quotient, defining the Whittaker model $\mathcal{W}(\sigma)$ in the latter case as in Section 3.3 above

For $W_0 \in \mathcal{W}(\pi)$, $\Phi_1 \in \mathcal{S}(F^2)$, and $W_2 \in \mathcal{W}(\sigma)$, we define

$$Z(W_0, \Phi_1, W_2; s) = \int_{Z_G N_H \backslash H} W_0(\iota(h)) f^{\Phi_1}\left(h_1; 1, (\chi_\pi \chi_\sigma)^{-1}, s\right) W_2(h_2)\, \mathrm{d}h.$$

Theorem 5.1 (Novodvorsky)**.** *There is $R < \infty$, depending on π and σ, such that the integral converges for $\Re(s) > R$ and has analytic continuation as a rational function in q^s. The $\mathbf{C}$-vector space spanned by the functions $Z(W_0, \Phi, W_2; s)$ for varying (W_0, Φ, W_2) is a fractional ideal of $\mathbf{C}[q^s, q^{-s}]$ containing the constant functions.* □

See [Nov79], [Sou84], and [LPSZ21, §8] for further details.

Definition 5.2. *We let $L^{\mathrm{Nov}}(\pi \times \sigma, s)$ be the unique L-factor generating the fractional ideal of values of the zeta integral.*

This is the L-factor featuring in Conjecture α. Although the conjecture is open in general, many cases can be obtained from the following result of Soudry. If τ_1, τ_2 are irreducible generic representations of $\mathrm{GL}(2, F)$ with the same central character, then we can regard the product $\tau_1 \boxtimes \tau_2$ as a representation of the group

$$(\mathrm{GL}(2, F) \times \mathrm{GL}(2, F)) / \{(z, z^{-1}) : z \in F^\times\}.$$

This group is isomorphic to the split orthogonal similitude group $\mathrm{GSO}(4, F)$, and there is a theta-lifting from this group to $\mathrm{GSp}(4, F)$. The non-supercuspidal generic representations that are θ-lifts from $\mathrm{GSO}(2, 2)$ are those of Sally–Tadić types I, IIa, Va, VIa, VIIIa, X and XIa, while types IIIa, IVa, VII and IXa are not in the image. The image of the θ-lift also contains some (but not all) of the generic supercuspidal representations of $\mathrm{GSp}(4)$.

Theorem 5.3 (Soudry, [Sou84])**.** *Suppose that π is an irreducible generic representation of the form $\pi = \theta(\tau_1, \tau_2)$, where τ_i are irreducible generic representations of $\mathrm{GL}(2, F)$ as above. Suppose that σ is irreducible, and if σ is supercuspidal, that it is not an unramified twist of $\tau_1^\vee$ or $\tau_2^\vee$. Then we have*

$$L^{\mathrm{Nov}}(\pi \times \sigma, s) = L(\pi \times \sigma, s) = L(\tau_1 \times \sigma, s) L(\tau_2 \times \sigma, s),$$

where $L(\tau_i \times \sigma, s)$ are the $\mathrm{GL}_2 \times \mathrm{GL}_2$ Rankin–Selberg L-factors. □

5.2. **An auxiliary integral.** To better understand Novodvorsky's integral, we write it in terms of the following auxiliary function:

Definition 5.4. *For $W_0 \in \mathcal{W}(\pi)$ and $W_2 \in \mathcal{W}(\sigma_2)$ we define*

$$Z(W_0, W_2, s) := \int_{N_2 \backslash \mathrm{GL}_2} W_0\left(\begin{pmatrix} \det g & & \\ & g & \\ & & 1 \end{pmatrix}\right) W_2(g) |\det g|^{s-1}\, \mathrm{d}g,$$

where N_2 is the upper-triangular unipotent subgroup of $\mathrm{GL}(2, F)$.

One computes that the function on h defined by $h \mapsto Z(hW_0, h_2W_2, s)$ depends only on the first projection h_1 of h, and belongs to the principal-series $\mathrm{GL}(2,F)$-representation $i_{1-s}(1, \nu^{-1})$, where $\nu = (\chi_\pi\chi_\sigma)^{-1}$.

Proposition 5.5. *For W_0, W_2 as above and $\Phi \in \mathcal{S}(F^2)$, we have*

$$Z(W_0, \Phi_1, W_2; s) = \left\langle Z(W_0, W_2; s), f^{\Phi}(1, \nu, s) \right\rangle,$$

where $\langle -,- \rangle$ denotes the canonical duality pairing between $i_{1-s}(1,\nu^{-1})$ and $i_s(1,\nu)$, given by integration over $B_2\backslash \mathrm{GL}_2$.

Proof. Let H_+ be the subgroup $\{(h_1, h_2) \in H : h_1 \text{ is upper-triangular}\}$ of H. Then $Z_G N_H \leqslant H_+$, and we can write the integral over $Z_G N_H \backslash H$ defining $Z(W_0, \Phi, W_2; s)$ as an integral over $Z_G N_H\backslash H_+$ composed with an integral over $H_+\backslash H$. However, the map $\mathrm{GL}(2,F) \to H_+$ given by $\gamma \mapsto \left(\begin{pmatrix} \det\gamma & \\ & 1\end{pmatrix}, \gamma\right)$ gives a bijection $Z_G N_H\backslash H_+ \cong N_2\backslash \mathrm{GL}_2$; and projection onto the first factor clearly identifies $H_+\backslash H$ with $B_2\backslash \mathrm{GL}_2$. □

5.3. Exceptional poles of the $\mathrm{GSp}(4)\times\mathrm{GL}(2)$ integral.

Definition 5.6. *We define $L^{\mathrm{Nov}}_{\mathrm{reg}}(\pi\times\sigma, s)$ to be the L-factor generating the fractional ideal*

$$\left\{Z(W_0, \Phi_1, W_2; s) : W_0 \in \mathcal{W}(\pi), \Phi_1 \in \mathcal{S}_0(F^2), W_2 \in \mathcal{W}(\sigma)\right\},$$

and we define $L^{\mathrm{Nov}}_{\mathrm{ex}}(\pi\times\sigma, s)$ to be the quotient, so that

$$L^{\mathrm{Nov}}(\pi\times\sigma, s) = L^{\mathrm{Nov}}_{\mathrm{reg}}(\pi\times\sigma, s)L^{\mathrm{Nov}}_{\mathrm{ex}}(\pi\times\sigma, s).$$

(We use implicitly here the fact that the fractional ideal ($\star$) contains the constant functions, which follows from the proof of [LPSZ21, Theorem 8.9(i)].)

Proposition 5.7. *The L-factor $L^{\mathrm{Nov}}_{\mathrm{reg}}(\pi\times\sigma, s)$ is also the L-factor generating the fractional ideal*

$$\left\{Z(W_0, W_2; s) : W_0 \in \mathcal{W}(\pi), W_2 \in \mathcal{W}(\sigma)\right\}.$$

Proof. This follows from the formula of Proposition 5.5, since the functions $f^{\Phi}(1,\nu,s)$ for $\Phi \in \mathcal{S}_0(F^2)$ are entire and span the whole of $i_s(1,\nu)$. □

Corollary 5.8. *The poles of $L^{\mathrm{Nov}}_{\mathrm{ex}}(\pi\times\sigma, s)$, if any, are simple. If $s = s_0$ is a pole of this factor, then we must have $\chi_\pi\chi_\sigma|\cdot|^{2s_0} = 1$, and*

$$\mathrm{Hom}_H\left(\pi \otimes (|\cdot|^{s_0} \boxtimes \sigma), \mathbf{C}\right) \ne 0.$$

Proof. It follows from the previous proposition that if $Z(W_0, \Phi, W_2; s)/L^{\mathrm{Nov}}_{\mathrm{reg}}(\pi\times\sigma, s)$ has a pole of order $n \geqslant 1$ at $s = s_0$, for some pen (W_0, Φ, W_2), then $f^{\Phi}(1,\nu,s)$ must also have a pole of order n at s_0 (where $\nu = (\chi_\pi\chi_\sigma)^{-1}$ as above). This can only occur if $n = 1$ and $|\cdot|^{2s_0} = \nu$. Moreover, since the residues of the f^{Φ} land in the one-dimensional representation $|\cdot|^{s_0}$, the residue at an exceptional pole defines a non-zero element of $\mathrm{Hom}_H\left(\pi \otimes (|\cdot|^{s_0} \boxtimes \sigma), \mathbf{C}\right)$. □

5.4. Regular poles.

We now relate $L^{\mathrm{Nov}}_{\mathrm{reg}}(\pi\times\sigma, s)$ to the supercuspidal support of π and σ. Recall that an irreducible G-representation π is said to have *supercuspidal support in P*, for a parabolic $P \subseteq G$, if it is a subquotient of the parabolic induction of a supercuspidal representation of the Levi of P. There are four conjugacy classes of parabolic subgroups in $G = \mathrm{GSp}(4,F)$: the whole group, the *Klingen* and *Siegel* parabolics

$$P_{\mathrm{Kl}} = \begin{pmatrix} \star & \star & \star & \star \\ & \star & \star & \star \\ & \star & \star & \star \\ & & & \star \end{pmatrix} \qquad \text{and} \qquad P_{\mathrm{Si}} = \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ & & \star & \star \\ & & \star & \star \end{pmatrix}$$

and the standard Borel $B_G = P_{\mathrm{Sieg}} \cap P_{\mathrm{Kl}}$.

Proposition 5.9. *For any W_0 and W_2, we have*

$$Z(W_0, W_2, s) = \int_{B_2\backslash \mathrm{GL}_2} Y(gW_0, gW_2, s)\, \mathrm{d}g,$$

where

$$Y(W_0, W_2, s) = \int_{F^\times\times F^\times} W_0\left(\begin{pmatrix} xy^2 & & & \\ & xy & & \\ & & y & \\ & & & 1\end{pmatrix}\right) W_2\left(\begin{pmatrix} x & \\ & 1\end{pmatrix}\right)\chi_\sigma(y)|x|^{s-2}|y|^{2s-2}\, \mathrm{d}^\times x\, \mathrm{d}^\times y.$$

Proof. This follows by writing B_2 as the semidirect product of N_2 and the maximal torus $T_2 \cong F^\times \times F^\times$. $\square$

Since $B_2\backslash \mathrm{GL}_2$ is compact, the fractional ideal of $\mathbf{C}[q^{\pm s}]$ generated by $Z(W_0, W_2, s)$ for all (W_0, W_2) is contained in that generated by the functions $Y(W_0, W_2, s)$. So we need to investigate the possible asymptotic behaviour of the function $(x, y) \mapsto W_0(\begin{pmatrix} xy^2 &&& \\ & xy && \\ && y & \\ &&& 1\end{pmatrix})W_2((\begin{smallmatrix} x & \\ & 1\end{smallmatrix}))$, for $W_0 \in \mathcal{W}(\pi)$ and $W_2 \in \mathcal{W}(\sigma)$. It follows from Lemma 2.6.2 of [RS07] that the support of this function is contained in a compact subset of $F \times F$, so the poles of the $Y(W_0, W_2, s)$, if any, arise from asymptotics as $x \to 0$ or $y \to 0$.

Proposition 5.10.

- *If π is supercuspidal, or its supercuspidal support lies in the Siegel parabolic, then the support of $y \mapsto W_0(\begin{pmatrix} y^2 &&& \\ & y && \\ && y & \\ &&& 1\end{pmatrix})$ is compact in $F^\times$, for all $W_0 \in \mathcal{W}(\pi)$.*
- *If π is supercuspidal, or its supercuspidal support lies in the Klingen parabolic, then the support of $x \mapsto W_0(\begin{pmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1\end{pmatrix})$ is compact in $F^\times$ for all $W_0 \in \mathcal{W}(\pi)$.*
- *If σ is supercuspidal, then the support of $x \mapsto W_2((\begin{smallmatrix} x & \\ & 1\end{smallmatrix}))$ is compact in $F^\times$, for any $W_2 \in \mathcal{W}(\sigma)$.*

Proof. We prove the first claim; the other two are similar. Let N_{Kl} denote the unipotent radical of P_{Kl}. The hypotheses imply that $J_{\mathrm{Kl}}(\pi) = 0$, where $J_{\mathrm{Kl}}(\pi)$ is the Jacquet functor. As a vector space $J_{\mathrm{Kl}}(\pi) = \pi/\pi(N_{\mathrm{Kl}})$, where $\pi(N_{\mathrm{Kl}})$ is the span of vectors of the form $(n-1)v$ for $v \in \pi$ and $n \in N_{\mathrm{Kl}}$. However, one computes easily using (1) that if $W_0 = (n-1)W_0'$ for some $W_0' \in \mathcal{W}(\pi)$ and $n \in N_{\mathrm{Kl}}$, then $W_0\left(\begin{pmatrix} y^2 &&& \\ & y && \\ && y & \\ &&& 1\end{pmatrix}\right) = (e(ty)-1)W_0'\left(\begin{pmatrix} y^2 &&& \\ & y && \\ && y & \\ &&& 1\end{pmatrix}\right)$, where $t \in F$ is the $(1,2)$-entry of n. If we choose y small enough, then $e(ty) = 1$; so for all such y we have $W_0\left(\begin{pmatrix} y^2 &&& \\ & y && \\ && y & \\ &&& 1\end{pmatrix}\right) = 0$. $\square$

Proposition 5.11. *Suppose that* either

- *π is supercuspidal,*
- *σ is supercuspidal, and π is not a subquotient of a representation induced from the Klingen parabolic of the form $\chi \rtimes \tau$, with τ an unramified twist of $\sigma^\vee$.*

Then $L^{\mathrm{Nov}}_{\mathrm{reg}}(\pi \times \sigma, s) = 1$, so all poles of $L^{\mathrm{Nov}}(\pi \times \sigma, s)$ are exceptional.

Proof. If π is supercuspidal, or σ is supercuspidal and π is supported in the Siegel parabolic, then the above results show that $W_0(\begin{pmatrix} xy^2 &&& \\ & xy && \\ && y & \\ &&& 1\end{pmatrix})W_2((\begin{smallmatrix} x & \\ & 1\end{smallmatrix}))$ has compact support for all (W_0, W_2), so the integrals $Y(W_0, W_2, s)$ have no poles, and hence the $Z(W_0, W_2, s)$ *a fortiori* have no poles either.

This leaves the more delicate case when σ is supercuspidal, and π is supported in the Klingen parabolic. The above arguments show that, if s_0 is a pole of $L^{\mathrm{Nov}}_{\mathrm{reg}}(\pi \times \sigma, s)$, then the leading term of $Z(W_0, W_2, s)$ at s_0 vanishes when $W_0 \in \mathcal{W}(\pi)(N_{\mathrm{Kl}})$. Hence the leading term depends only on the image of W_0 in the Klingen Jacquet module of π; and this leading term defines a non-zero linear functional on $J_{\mathrm{Kl}}(\pi) \otimes \sigma$ which is $\mathrm{GL}(2, F)$-equivariant, up to an unramified twist, where we regard $\mathrm{GL}(2, F)$ as a subgroup of the Klingen Levi $F^\times \times \mathrm{GL}(2, F)$. Hence some unramified twist of $\sigma^\vee$ appears in the Jacquet module, and the result follows. $\square$

6. Relating the zeta integrals

We'll fix throughout this section a generic irreducible representation π of G.

6.1. **The basic formula.** The following is Proposition 8.4 of [LPSZ21]:

Proposition 6.1. *For any smooth characters μ_2, ν_2 of F, we have*

$$Z(W_0, \Phi_1, W^{\Phi_2}(\mu_2, \nu_2); s) = L(\pi \times \nu_2, s) Z(\widetilde{B}_{W_0}, \Phi_1, \Phi_2; \Lambda, \mu_2, s),$$

where Λ is the character $\left(\chi_\pi \nu_2|\cdot|^{s-\frac{1}{2}}, \nu_2^{-1}|\cdot|^{\frac{1}{2}-s}\right)$, and $\widetilde{B}_{W_0} = \widetilde{B}_{W_0}(g; \nu_2, s) \in \mathcal{B}_\Lambda(\pi)$.

Here $W^{\Phi_2}(-; \mu_2, \nu_2)$ is the Whittaker function defined in Section 3.3.

Corollary 6.2. *If $\sigma = i(\mu_2, \nu_2)$ is a principal-series representation with $\mu_2/\nu_2 \neq |\cdot|^{-1}$, then we have*

$$L^{\mathrm{Nov}}(\pi \times \sigma, s) = L(\pi \times \mu_2, s) L(\pi \times \nu_2, s).$$

Proof. Since the functions $W^{\Phi_2}(-; \mu_2, \nu_2)$ for varying Φ_2 form the Whittaker model $\mathcal{W}(\sigma)$, the L-factor $L^{\mathrm{Nov}}(\pi \times \sigma, s)$ is the unique L-factor generating the fractional ideal $\{Z(W_0, \Phi_1, W^{\Phi_2}(\mu_2, \nu_2); s) : W_0 \in \mathcal{W}(\pi), \Phi_1, \Phi_2 \in \mathcal{S}(F^2)\}$. On the other hand, the map $W_0 \mapsto \widetilde{B}_{W_0}$ is an isomorphism $\mathcal{W}(\pi) \cong \mathcal{B}_\Lambda(\pi)$, so the fractional ideal $\{Z(\widetilde{B}_{W_0}, \Phi_1, \Phi_2; \Lambda, \mu_2, s) : W_0 \in \mathcal{W}(\pi), \Phi_1, \Phi_2 \in \mathcal{S}(F^2)\}$ is generated by $L(\pi \times \mu_2, s)$ by Theorem 4.2. ∎

In particular, this shows that Conjecture α holds if σ is an irreducible principal series (this is Theorem 8.9(i) of [LPSZ21]); and we have chosen our definition of $\mathcal{W}(\sigma)$, when σ is a reducible principal series, in order to make the same statement also be valid in the reducible case.

6.2. Exceptional poles: the principal-series case.

Proposition 6.3. *Suppose $\sigma = i(\mu_2, \nu_2)$ with $\mu_2/\nu_2 \neq |\cdot|^{\pm 1}$, so σ is an irreducible principal series.*

For $s_0 \in \mathbf{C}$, we have $\chi_\pi \chi_\sigma |\cdot|^{2s_0} = 1$ if and only if $L(\lambda_1 \mu_2, s + \frac{1}{2})$ has a pole at $s = s_0$, where $(\lambda_1, \lambda_2) = \left(\chi_\pi \nu_2 |\cdot|^{s_0 - \frac{1}{2}}, \nu_2^{-1}|\cdot|^{\frac{1}{2}-s_0}\right)$ as above. If this condition is satisfied, then $s = s_0$ is an exceptional pole of $L^{\mathrm{Nov}}(\pi \times \sigma, s)$ if and only if it is a subregular pole of $L(\pi \times \mu_2, s)$.

Proof. This is clear from the same argument as Corollary 6.2. ∎

6.3. **Exceptional poles: the Steinberg case.** We now consider the formula of Proposition 6.1 with $\mu_2 = 1$ and $\nu_2 = |\cdot|$, so that $\sigma = i(\mu_2, \nu_2)$ is reducible with 1-dimensional subrepresentation, and its unique irreducible quotient is the twist $\mathrm{St} \otimes |\cdot|^{1/2}$ of the Steinberg representation. We write W^{Φ_2} for $W^{\Phi_2}(\mu_2, \nu_2)$; hence the space of functions W^{Φ_2} for $\Phi \in \mathcal{S}(F^2)$ is the Whittaker model of $\sigma' = i(\nu_2, \mu_2)$, and the W^{Φ_2} with $\Phi \in \mathcal{S}_0(F^2)$ is the Whittaker model of $\mathrm{St} \otimes |\cdot|^{1/2}$.

We are interested in the following three fractional ideals of $\mathbf{C}[q^s, q^{-s}]$:

$$I := \left(\frac{Z(W_0, \Phi_1, W^{\Phi_2}; s)}{L(\pi, s) L(\pi, s+1)} : W_0 \in \mathcal{W}(\pi), \Phi_1 \in \mathcal{S}(F^2), \Phi_2 \in \mathcal{S}(F^2) \right)$$

$$J := \left(\frac{Z(W_0, \Phi_1, W^{\Phi_2}; s)}{L(\pi, s) L(\pi, s+1)} : W_0 \in \mathcal{W}(\pi), \Phi_1 \in \mathcal{S}(F^2), \Phi_2 \in \mathcal{S}_0(F^2) \right)$$

$$K := \left(\frac{Z(W_0, \Phi_1, W^{\Phi_2}; s)}{L(\pi, s) L(\pi, s+1)} : W_0 \in \mathcal{W}(\pi), \Phi_1 \in \mathcal{S}_0(F^2), \Phi_2 \in \mathcal{S}_0(F^2) \right)$$

Corollary 6.2 shows that I is the unit ideal. On the other hand, from the definitions of the $\mathrm{GSp}_4 \times \mathrm{GL}_2$ L-factors, we have

$$J = \left(\frac{L^{\mathrm{Nov}}(\pi \times \mathrm{St}, s + \frac{1}{2})}{L(\pi, s) L(\pi, s+1)} \right), \quad K = \left(\frac{L^{\mathrm{Nov}}_{\mathrm{reg}}(\pi \times \mathrm{St}, s + \frac{1}{2})}{L(\pi, s) L(\pi, s+1)} \right).$$

Since clearly $I \supseteq J \supseteq K$, we see that J and K are integral ideals (not just fractional ideals) of $\mathbf{C}[q^{\pm s}]$.

Proposition 6.4. *The ideal K vanishes at s_0 if and only if s_0 is a subregular pole of $L(\pi, s)$ (in the sense of Definition 4.8).*

Proof. This follows from Proposition 6.1, together with the definition of subregular poles. ∎

Remark 6.5. It is *not* true that the order of vanishing of K at s_0 coincides with the order of the pole of $L^{\Lambda}_{\mathrm{sub}}(\pi, s)$ at $s = s_0$, where Λ is the Bessel character $\left(|\cdot|^{-1/2-s_0}, \chi_\pi |\cdot|^{1/2+s_0}\right)$. The order of pole of $L^{\Lambda}_{\mathrm{sub}}(\pi, s)$ is always either 0 or 1, as we have seen; but the orders of vanishing of J and K can be > 1 in some cases. (This difference arises because L_{sub} detects the infinitesimal behaviour of Piatetski-Shapiro's integrals as s varies for a fixed Λ, but the ideals J and K detect the behaviour along a one-parameter family in which s and Λ both vary.) $\diamond$

Corollary 6.6. *If $s_0 \in \mathbf{C}$ is such that $\chi_\pi |\cdot|^{2s_0+1} \neq 1$, then s_0 is a subregular pole of $L(\pi, s)$ if and only if it is a pole of the ratio* $\dfrac{L(\pi,s)L(\pi,s+1)}{L^{\mathrm{Nov}}\left(\pi \times \mathrm{St}, s+\frac{1}{2}\right)}$.

Proof. If $\chi_\pi |\cdot|^{2s_0+1} \neq 1$, then s_0 cannot be a pole of $L^{\mathrm{Nov}}_{\mathrm{ex}}(\pi \times \mathrm{St}, s + \frac{1}{2})$. So the orders of vanishing of J and K at $s = s_0$ are the same, and the result follows from the previous proposition. $\square$

Proposition 6.7. *Suppose $\chi_\pi |\cdot|^{2s_0+1} = 1$. Then J does not vanish identically at $s = s_0$. Hence $s = s_0$ is a subregular pole if and only if it is a pole of $L^{\mathrm{Nov}}_{\mathrm{ex}}(\pi \times \mathrm{St}, s + \frac{1}{2})$.*

Proof. The symmetry condition on s_0 shows that if J vanishes identically, then the same is true if we interchange Φ_1 and Φ_2. Hence $\frac{Z(W_0,\Phi_1,W^{\Phi_2};s)}{L(\pi,s)L(\pi,s+1)}$ in fact vanishes for all Φ_1, Φ_2 satisfying $\Phi_1(0,0)\Phi_2(0,0) = 0$. This shows that s_0 is an exceptional pole of the Piatetski-Shapiro L-factor, and such poles cannot occur for generic representations as we have seen above. $\square$

Note that Proposition 6.7 shows that part (1) of Theorem C is true, assuming Theorem A. Similarly, Corollary 6.6 shows that conditions (i) and (ii) of Theorem C are equivalent.

7. Compatibility with the Langlands parameters

7.1. **Langlands parameters.** Let ρ be a Frobenius-semisimple Weil–Deligne representation $\mathrm{WD}(F) \to \mathrm{GL}(n, \mathbf{C})$. Then we can write ρ (uniquely up to isomorphism) in the form

$$\rho = \bigoplus_i \rho_i \otimes \mathrm{sp}(n_i),$$

where $n_i \geqslant 1$ are integers and ρ_i are irreducible representations of the Weil group (with trivial monodromy action), such that $\sum_i n_i \dim(\rho_i) = n$. Here $\mathrm{sp}(j)$ denotes the $(j-1)$-st symmetric power of the Langlands parameter of the Steinberg representation of GL_2, which is the 2-dimensional representation with Frobenius acting as $\begin{pmatrix} q^{-1/2} & \\ & q^{1/2} \end{pmatrix}$ and monodromy as $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$. Note that we have

$$L(\rho, s) = \prod_i L(\rho_i, s + \tfrac{n_i - 1}{2}).$$

Lemma 7.1. *With the above notations, we have*

$$\frac{L(\rho,s)L(\rho,s+1)}{L(\rho \times \mathrm{sp}(2), s+\frac{1}{2})} = \prod_{\{i : n_i = 1\}} L(\rho_i, s),$$

and similarly

$$\frac{L(\rho \otimes \mathrm{sp}(2), s)L(\rho \otimes \mathrm{sp}(2), s+1)}{L\left(\rho \otimes \mathrm{sp}(2) \otimes \mathrm{sp}(2), s+\frac{1}{2}\right)} = \prod_{\{i : n_i = 2\}} L(\rho_i, s).$$

Proof. This is a straightforward computation using the fact that

$$\mathrm{sp}(n) \otimes \mathrm{sp}(2) = \begin{cases} \mathrm{sp}(n+1) \oplus \mathrm{sp}(n-1) & \text{if } n \geqslant 2, \\ \mathrm{sp}(2) & \text{if } n = 1. \end{cases}$$ $\square$

We shall apply this to the 4-dimensional representations arising from the local Langlands correspondence for G [GT11]; we write ϕ_π for the Langlands parameter of π, which we consider as a 4-dimensional Weil–Deligne representations by composing with the inclusion $\mathrm{GSp}(4, \mathbf{C}) \hookrightarrow \mathrm{GL}(4, \mathbf{C})$. We also have the local Langlands correspondence $\sigma \mapsto \phi_\sigma$ for $\mathrm{GL}(2, F)$. We refer to [RS07, §2.4] for an explicit description of ϕ_π for non-supercuspidal π.

Proposition 7.2. *If π is supercuspidal, or if σ is supercuspidal and π is not a subquotient of the Klingen parabolic induction of an unramified twist of $\sigma^\vee$, then Conjecture α implies Conjecture β.*

Proof. I claim that under these hypotheses, the Langlands L-factor $L(\pi \times \sigma, s)$ has at most simple poles, and these all arise from one-dimensional summands of $\phi_\pi \otimes \phi_\sigma$.

This claim implies the proposition, since (assuming Conjecture α), Conjecture β in this case amounts to the assertion that all poles of the Novodvorsky L-factor are exceptional, which is true by Proposition 5.11.

Let us now prove the claim. First, we suppose σ is supercuspidal. In this case, ϕ_σ is an irreducible 2-dimensional representation of the Weil group (with trivial monodromy action). If $L(\pi \times \sigma, s)$ has any poles, then ϕ_π must have one or more direct summands isomorphic to unramified twists of $\phi_\sigma^\vee \otimes \mathrm{sp}(j)$, for some j. However, if there is a summand with $j > 1$, or more than one such summand, then this implies that π is a subquotient of the induction of some twist of $\sigma^\vee$ (using the explicit description of the Langlands correspondence for non-supercuspidal representations described in §2.4 of [RS07]), contradicting our assumptions. In the remaining case, when there is precisely one such summand and it has $j = 1$, the corresponding summand of the tensor product also has trivial monodromy, as required.

Now let us suppose π is supercuspidal. Then ϕ_π is either irreducible of dimension 4, or is the direct sum of two *distinct* 2-dimensional irreducible representations (with the same determinant). So the L-factor is trivial unless σ is also supercuspidal, and we may argue as before. $\square$

7.2. **Proof of Theorem A for Steinberg σ.** The results of the previous section give a complete characterisation of the poles of the ratio $\frac{L(\pi,\, s)L(\pi,\, s+1)}{L^{\mathrm{Nov}}\left(\pi\times\mathrm{St},\, s+\frac{1}{2}\right)}$: they are precisely the complex numbers s_0 such that $\chi_\pi|\cdot|^{2s_0+1} \ne 1$ and $L(\pi, s)$ has a subregular pole. We shall use this, together with the tables of subregular poles in [RW17, RW18], to compute $L^{\mathrm{Nov}}(\pi \times \mathrm{St}, s)$, and hence prove Theorem A of the introduction.

Theorem 7.3 (Theorem A)**.** *Let π be a generic irreducible representation of* $\mathrm{GSp}(4, F)$*. Then Conjecture α holds for σ the Steinberg representation, i.e. we have*

$$L^{\mathrm{Nov}}(\pi \times \mathrm{St}, s) = L(\pi \times \mathrm{St}, s).$$

Proof. We can assume that π is either supercuspidal, or that its Sally–Tadić type is one of {IIIa, IVa, VII, IXa}, since Conjecture α is already known in the remaining cases by Theorem 5.3.

According to Theorem 4.9, each of these classes of representations has the property that $L(\pi, s)$ has no subregular poles. For IIIa and IVa, there may be poles, but they are never subregular; for VII, IXa and supercuspidals, there are no poles at all. So for these representations, we have $L^{\mathrm{Nov}}(\pi \times \mathrm{St}, s) = L(\pi, s - \frac{1}{2})L(\pi, s + \frac{1}{2})$. On the other hand, since the Langlands parameters of these representations have no 1-dimensional summands, we have $L(\pi \times \mathrm{St}, s) = L(\pi, s - \frac{1}{2})L(\pi, s + \frac{1}{2})$ by Lemma 7.1. So Conjecture α holds for all these representations. $\square$

8. Proof of Theorems B, C and D

Proof of Theorem C. Let π and s_0 be as in the theorem. If $\chi_\pi|\cdot|^{2s_0+1} \ne 1$, then Corollary 6.6 shows that s_0 is an exceptional pole of $L(\pi, s)$ if and only if it is a pole of $\frac{L(\pi,s)L(\pi,s+1)}{L^{\mathrm{Nov}}(\pi\times\mathrm{St},s+\frac{1}{2})}$. By Theorem A, which we have just proved, the denominator agrees with the Langlands L-factor $L(\pi \times \mathrm{St}, s + \frac{1}{2})$. This completes the proof of Theorem C when $\chi_\pi|\cdot|^{2s_0+1} \ne 1$.

If $\chi_\pi|\cdot|^{2s_0+1} = 1$, then Proposition 6.7 (combined with Theorem A) shows that s_0 is not a pole of $\frac{L(\pi,s)L(\pi,s+1)}{L(\pi\times\mathrm{St},s+\frac{1}{2})}$. So we must check that s_0 is a subregular pole if and only if ϕ_π has a direct summand of the form $|\cdot|^{-(s_0+1/2)} \otimes \mathrm{sp}(2)$. This follows by a case-by-case check from Theorem 4.9 combined with the tables of Langlands parameters in [RS07]. $\square$

Proof of Theorem B. We first suppose σ is an irreducible principal series $i(\mu_2, \nu_2)$. Twisting π appropriately, we may assume $\mu_2 = 1$; and the irreducibility gives $\nu_2 \ne |\cdot|^{\pm 1}$. Moreover, s_0 is such that $\chi_\pi \nu_2|\cdot|^{2s_0} = 1$, and we may assume $s_0 = 0$.

By Proposition 6.3, 0 is an exceptional pole of the Novodvorsky L-factor if and only if it is a subregular pole of $L(\pi, s)$. Moreover, the irreducibility of σ shows that $\nu_2 \neq |\cdot|$, so $\chi_\pi|\cdot|^{2s_0+1} = \nu_2^{-1}|\cdot| \neq 1$. So, by the first case of Theorem C, 0 is an exceptional pole of $L(\pi \times \sigma, s)$ if and only if ϕ_π has a 1-dimensional trivial summand; and this in turn implies that $\phi_\pi \otimes \phi_\sigma$ also has such a summand, since $\phi_\pi \otimes \phi_\sigma = \phi_\pi \oplus \phi_{\pi\otimes\nu}$.

Conversely, if $\phi_\pi \otimes \phi_\sigma$ has a trivial summand, then it must come from either ϕ_π or $\phi_{\pi\otimes\nu}$. If the former holds, then reversing the argument shows that $L(\pi \times \sigma, s)$ has an exceptional pole at 0. However, since $\nu = \chi_\pi^{-1}$, the two factors are dual to each other, so $\phi_{\pi\otimes\nu}$ has a trivial summand if and only if ϕ_π does.

We now suppose σ is a special representation. Again, we may assume $\sigma = \mathrm{St} \otimes |\cdot|^{1/2}$, so we are now in the case $\chi_\pi|\cdot|^{2s_0+1} = 0$. By Proposition 6.7, s_0 is an exceptional pole of $L(\pi \times \sigma, s)$ if and only if it is a subregular pole of $L(\pi, s)$; and the second case of Theorem C shows that this occurs if and only if s_0 is a pole of the L-factor of a 2-dimensional summand of ϕ_π of the form $|\cdot|^{-(s_0+1/2)} \otimes \mathrm{sp}(2)$. Since ϕ_π cannot have any 3-dimensional summands, there is a bijection between 2-dimensional summands of ϕ_π and 1-dimensional summands of $\phi_\pi \otimes \phi_\sigma$, sending $\rho \otimes \mathrm{sp}(2)$ to $\rho|\cdot|^{1/2} \otimes \mathrm{sp}(1)$. So we conclude that s_0 is an exceptional pole of $L(\pi \times \sigma, s)$ if and only if $\phi_\pi \otimes \phi_\sigma$ has a summand $|\cdot|^{-s} \otimes \mathrm{sp}(1)$. ∎

Proof of Theorem D for non-supercuspidal σ. Suppose first that $\sigma = i(\mu, \nu)$ is an irreducible principal series representation. Twisting π and σ appropriately, we may assume that $s_0 = 0$, so $\mu\nu = \chi_\pi^{-1}$.

Then we have

$$\mathrm{Hom}_H(\pi \otimes (\mathbb{1} \boxtimes \sigma), \mathbf{C}) \cong \mathrm{Hom}_H(\pi \otimes (\sigma \boxtimes \mathbb{1}), \mathbf{C}) = \mathrm{Hom}_H(\pi, \sigma^\vee \boxtimes \mathbb{1}) = \mathrm{Hom}_{H_+}(\pi, \rho)$$

where H_+ denotes the subgroup $((\begin{smallmatrix} \star & \star \\ 0 & \star \end{smallmatrix}), \star)$ of H, and ρ the character $((\begin{smallmatrix} a & \star \\ 0 & d \end{smallmatrix}), \star) \mapsto |a/d|^{1/2}\mu^{-1}(a)\nu^{-1}(d)$. Our claim is that this space is non-zero if and only if $L(\pi \times \sigma, s)$ has an exceptional pole at 0; by Proposition 6.3, the latter is equivalent to $L(\pi \times \mu, s)$ having a subregular pole at 0.

Similarly, if σ is the Steinberg representation and $\chi_\pi = 1$, then the natural map

$$\mathrm{Hom}_H(\pi, \mathrm{St} \boxtimes \mathbb{1}) \to \mathrm{Hom}_H(\pi, \Sigma \boxtimes \mathbb{1})$$

is an isomorphism, by [PS97, Theorem 4.3]. Again, the right-hand side can be interpreted as a space of H_+-invariant functionals, where we take ρ the character $((\begin{smallmatrix} a & \star \\ 0 & d \end{smallmatrix}), \star) \mapsto |a/d|$; and we want to show that this space is non-zero if and only if $L(\pi \times \mathrm{St}, s)$ has an exceptional pole at $s = 0$, which is equivalent to $L(\pi, s)$ having a subregular pole at $-\frac{1}{2}$, by Proposition 6.7.

Following §4 of [RW18], we refer to elements of $\mathrm{Hom}_{H_+}(\pi, \rho)$, where ρ is a character of H_+, as "(H_+, ρ)-functionals". The claim we need to prove is:

> Let ρ be the character $((\begin{smallmatrix} a & \star \\ 0 & d \end{smallmatrix}), \star) \mapsto |a/d|^{1/2}\mu^{-1}(a)\nu^{-1}(d)$ of H_+, where μ, ν are characters of $F^\times$ such that $\mu\nu = \chi_\pi^{-1}$. Then the space of (H_+, ρ)-functionals on π is 1-dimensional if $L(\pi \times \mu, s)$ has a subregular pole at $s = 0$, and zero otherwise.

This follows from the results of [RW18, §5]. ∎

9. Proof of Theorem E

9.1. **Uniqueness for** $\mathrm{GSp}(4) \times \mathrm{GL}(2)$**.** Let π, σ be irreducible generic representations of $\mathrm{GSp}(4, F)$ and $\mathrm{GL}(2, F)$ respectively. Then, for any $s_0 \in \mathbf{C}$, the map $\tilde{Z}_{s_0} : \mathcal{W}(\pi) \otimes \mathcal{S}(F^2) \otimes \mathcal{W}(\sigma) \to \mathbf{C}$ defined by

$$(W_0, \Phi_1, W_2) \mapsto \left.\frac{Z(W_0, \Phi_1, W_2, s)}{L^{\mathrm{Nov}}(\pi \times \sigma, s)}\right|_{s=s_0}$$

satisfies $\tilde{Z}_{s_0}(hW_0, h_1\Phi_1, h_2W_2) = |\det h|^{-s_0}\tilde{Z}_{s_0}(W_0, \Phi_1, W_2)$. In particular, it factors through the maximal quotient of $\mathcal{S}(F^2)$ on which $F^\times$ acts via $\nu|\cdot|^{-2s_0}$, where $\nu = (\chi_\pi\chi_\sigma)^{-1}$. We are interested in the case $s_0 = 0$, $\nu = 1$, in which case this quotient is isomorphic to $\Sigma = i(|\cdot|^{1/2}, |\cdot|^{-1/2})$, via $\Phi \mapsto F^\Phi$. Thus we have $\tilde{Z}_{s_0}(W_0, \Phi_1, W_2) = \mathfrak{z}(W_0, F^{\Phi_1}, W_2)$ for some non-zero element $\mathfrak{z} \in \mathrm{Hom}_H(\pi \otimes (\Sigma \boxtimes \sigma), \mathbf{C})$.

There is a left-exact sequence

$$0 \to \mathrm{Hom}_H\Big(\pi \otimes (\mathbb{1} \boxtimes \sigma), \mathbf{C}\Big) \xrightarrow{\alpha} \mathrm{Hom}_H\Big(\pi \otimes (\Sigma \boxtimes \sigma), \mathbf{C}\Big) \xrightarrow{\beta} \mathrm{Hom}_H\Big(\pi \otimes (\mathrm{St} \boxtimes \sigma), \mathbf{C}\Big)$$

in which the first and third terms both have dimension $\leqslant 1$, by the multiplicity-one results for GSpin groups proved in [ET23] and the isomorphisms $G(F) \cong \mathrm{GSpin}(5)$ and $H \cong \mathrm{GSpin}(4)$. Conjecture ε(a) asserts that the middle group in the above sequence is always 1-dimensional, so the element $\mathfrak{z}$ is a basis.

Remark 9.1. Note that there do exist examples in which the first and last terms are both nonzero – one can construct such examples with π and σ principal-series. ◇

Proposition 9.2. *The element $\mathfrak{z}$ is in the image of α if and only if $s = 0$ is an exceptional pole of $L^{\mathrm{Nov}}(\pi \times \sigma, s)$.*

Proof. This is essentially a restatement of the definitions, since the F^{Φ} with $\Phi(0,0) = 0$ span the generic subrepresentation $\mathrm{St} \subset \Sigma$. □

If σ is non-supercuspidal, and $s = 0$ is not an exceptional pole of the Novodvorsky L-factor, then we know from Theorem D that in fact $\mathrm{Hom}_H(\pi \otimes (\mathbb{1} \boxtimes \sigma), \mathbf{C}) = 0$; so Conjecture ε(a) follows in this case (that is, we have proved Theorem E(a)(ii)). Conversely, if we assume Conjecture ε(a), then it follows that $\mathrm{Hom}_H(\pi \otimes (\mathbb{1} \boxtimes \sigma), \mathbf{C})$ is non-zero if and only if $\mathfrak{z}$ is in the image of α, so Conjecture ε(a) implies Conjecture δ.

9.2. **Proof of Theorem E(a)(i).** We now prove Theorem E in the case where $\chi_\pi = \tau^2$ for some smooth character τ. Replacing π and σ with the twists $\pi \times \tau$ and $\sigma \times \tau^{-1}$, which does not change either the Hom-space or the zeta-integral, we may in fact suppose that $\chi_\pi = 1$. In this case we can regard π as a representation of $G/Z_G = \mathrm{PGSp}(4, F) \cong \mathrm{SO}(5, F)$, and $\Sigma \boxtimes \sigma$ as a representation of the subgroup $H/Z_G \cong \mathrm{SO}(4, F)$.

We now apply the results of [MW12] on branching laws for representations of special orthogonal groups. In *op.cit.* a branching multiplicity $m(\sigma, (\sigma')^\vee)$ is defined for irreducible representations σ of $\mathrm{SO}(d, F)$ and σ' of $\mathrm{SO}(d', F)$, where $d > d'$ are any integers of differing parity. (The results of *op.cit.* also cover non-split special orthogonal groups as well, but we do not need this here.) If $d = d' + 1$, then $m(\sigma, (\sigma')^\vee)$ is just $\dim \mathrm{Hom}_{\mathrm{SO}(d',F)}(\sigma, (\sigma')^\vee) = \dim \mathrm{Hom}_{\mathrm{SO}(d',F)}(\sigma \otimes \sigma', \mathbf{C})$; in the other extreme case, if $d' = 0$, then $m(\sigma, (\sigma')^\vee)$ is the space of Whittaker functionals on σ.

The Proposition stated in Section 1.3 of [MW12] analyses these multiplicities when σ and σ' are (possibly reducible) parabolic inductions, in which case $m(\sigma, (\sigma')^\vee)$ still makes sense. For these results, suppose that σ is induced from a representation $\pi_1|\cdot|^{b_1} \times \cdots \times \pi_t|\cdot|^{b_t} \times \sigma_0$ of the Levi subgroup $\mathrm{GL}(d_1, F) \times \cdots \times \mathrm{GL}(d_t, F) \times \mathrm{SO}(d_0, F)$ of $\mathrm{SO}(d, F)$, where $d = 2(d_1 + \cdots + d_t) + d_0$, π_i is a tempered irreducible representation of $\mathrm{GL}(d_i, F)$, σ_0 is a tempered irreducible representation of $\mathrm{SO}(d_0, F)$, and $b_1 \geqslant \ldots \geqslant b_t \geqslant 0$ are real numbers. (The case $d_0 = 0$ or 1 is allowed, in which case we understand $\mathrm{SO}(d_0)$ to be the trivial group.) We also make the same assumptions *mutatis mutandis* for σ'. The Proposition stated in §1.3 of [MW12] (and proved in §1.3–1.8 of *op.cit.*) shows that $m(\sigma, (\sigma')^\vee)$ is given by $m(\sigma_0, (\sigma_0')^\vee)$ if $d_0 > d_0'$, or $m(\sigma_0', (\sigma_0)^\vee)$ if $d_0 < d_0'$; in particular, since these numbers are known to be $\leqslant 1$ (by the results quoted in the introduction of *op.cit.*), we have $m(\sigma, (\sigma')^\vee) \leqslant 1$.

This class of parabolically-induced representations includes all generic irreducible representations; but it also contains some reducible representations – crucially, the reducible representations of $\mathrm{SO}(4, F)$ we are calling $\Sigma \boxtimes \sigma$, for any generic irreducible representation of $\mathrm{SO}(3, F) \cong \mathrm{PGL}(2, F)$, or $\Sigma \boxtimes \Sigma$, both have this form. Hence, applying this result with $d = 5$, $d' = 4$, and the σ and σ' of *op.cit.* taken to be our π and $\Sigma \boxtimes \sigma$, we have $\dim \mathrm{Hom}_{\mathrm{SO}(4,F)}(\pi \otimes (\Sigma \boxtimes \sigma), \mathbf{C}) \leqslant 1$ as required.

9.3. **Uniqueness for $\mathrm{GSp}(4)$.** We also have a slight strengthening of the above result in the case when σ is itself a twist of the Steinberg representation. Via twisting, we shall take $s_0 = 0$ and χ_π trivial, and consider the space $\mathrm{Hom}_H(\pi \otimes (\Sigma \boxtimes \Sigma), \mathbf{C})$. The argument of Moeglin–Waldspurger quoted above also applies in this situation, showing that shows that this space always has dimension 1.

Let us write $\Xi = \Sigma \boxtimes \Sigma$, and filter it as $\Xi_{00} \subset \Xi_0 \subset \Xi$ where $\Xi_{00} = \mathrm{St} \boxtimes \mathrm{St}$, $\Xi_0/\Xi_{00} = (\mathrm{St} \boxtimes \mathbb{1}) \oplus (\mathbb{1} \boxtimes \mathrm{St})$ and $\Xi/\Xi_0 = \mathbb{1} \boxtimes \mathbb{1}$.

Proposition 9.3. *The space $\mathrm{Hom}_H(\pi \otimes \Xi, \mathbf{C})$ contains a canonical non-zero homomorphism $\mathfrak{z}$ satisfying*

$$\mathfrak{z}(W_0, F^{\Phi_1}, F^{\Phi_2}) = \left. \frac{Z(\widetilde{B}_{W_0}, \Phi_1, \Phi_2; \Lambda, s)}{L(\pi, s)} \right|_{s=-1/2}, \qquad \Lambda = (1, 1).$$

Its restriction to $\pi \otimes \Xi_{00}$ is non-trivial if and only if $s = -\frac{1}{2}$ is not a subregular pole of $L(\pi, s)$, in which case $\mathrm{Hom}_H(\pi \otimes \Xi, \mathbf{C})$ is 1-dimensional spanned by $\mathfrak{z}$, and every non-generic subquotient ξ of Ξ satisfies $\mathrm{Hom}_H(\pi \otimes \xi, \mathbf{C}) = 0$.

Proof. One checks easily that the zeta-integral $Z(\tilde{B}_{W_0}, \dots)$ depends only on the image of Φ_i in the $F^\times$-coinvariants, or equivalently on F^{Φ_i}. Moreover, the fact that $\mathfrak{z}$ restricts non-trivially to Ξ_0 is precisely [PS97, Theorem 4.3]; and its proof moreover shows that $\mathrm{Hom}_H(\pi, \mathbf{C}) = 0$ for generic π.

If $s = -\frac{1}{2}$ is not a subregular pole, then Theorem D shows that $\mathrm{Hom}_H(\pi \otimes (\mathbb{1} \boxtimes \mathrm{St}), \mathbf{C})$ and $\mathrm{Hom}_H(\pi \otimes (\mathrm{St} \boxtimes \mathbb{1}), \mathbf{C})$ are zero. Hence the restriction map $\mathrm{Hom}_H(\pi \otimes \Xi, \mathbf{C}) \to \mathrm{Hom}_H(\pi \otimes \Xi_{00}, \mathbf{C})$ is injective. Since the latter space has dimension $\leqslant 1$ by [Wal12] the result follows. □

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